MESON THEORY OF NUCLEAR FORCES

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PREFACE TO THE SECOND EDITION

Since the publication of the last edition some of the older experiments (particularly those on angular distribution in proton–neutron scattering) have been rendered obsolete by more recent work. In addition, some consequences of the strong coupling theory have been clarified. In this respect, improvements have been introduced into the text.

On the other hand, the provisional state of the meson theory has become still more obvious as a result of the experimental discovery of at least two kinds of mesons by C. F. Powell and G. P. S. Occhialini, and by the absence of negative meson capture by the lighter nuclei by M. Conversi, E. Pancini, and O. Piccioni. In view of the failure of all present theories in explaining these new facts, the author saw no possibility of making an essential improvement in the substance of this book. But the recent success of C. M. G. Lattes and E. Gardner in producing mesons artificially will presumably bring forth a great change of the whole situation in the near future.

Zurich, Switzerland
June, 1948

WOLFGANG PAULI

Preface to the First Edition

The purpose of this volume is to make accessible to a larger number of readers the lectures which I gave in the autumn of 1944 at the Massachusetts Institute of Technology. Without pretending to contain anything essentially new, they may serve to give to students and research workers a first orientation in the more recent literature on the theory of the interaction of mesons with protons and neutrons (nucleons) and the interactions between nucleons derived from it. Despite the imperfections of my lectures, the original notes written by Dr. J. F. Carlson and Dr. A. J. F. Siegert have been amended only slightly, to preserve the informal character of the lectures and to emphasize the very provisional state of the problems in question, to which new experiments may in the future make important contributions. Special thanks are extended to Dr. S. T. Ma for reading and checking the proofs.

Princeton, New Jersey
February, 1946

WOLFGANG PAULI

CONTENTS

CHAPTER I

The meson theory of nuclear forces has been developed in analogy to the theory of the Coulomb force, which arises from the interaction of charged particles with the electromagnetic field. The short range of the nuclear forces is obtained by letting the particles corresponding to the photons have a rest mass,[1] the range of forces arising from a field of particles of rest mass μ being $\hbar/\mu c$. From experiments on scattering of neutrons by protons the range of nuclear forces is known to be $\sim 2 \times 10^{-13}$ cm, which leads to a rest mass $\mu \sim 200$ electron masses. Charged particles of this rest mass have subsequently been discovered in cosmic rays.

Another difference between the meson field and the electromagnetic field lies in the fact that mesons have charge and spin.

The main experimental data to be explained by the meson theory of nuclear forces are:

(*a*) the binding energy of the deuteron: $E_0 = 2.17$ Mev,

(*b*) the cross section for neutron–proton[2] scattering: $\sigma = (21 \pm 0.7) \times 10^{-24}$ cm^2,

(*c*) the quadrupole moment of the deuteron: $Q = 2.73 \times 10^{-27}$ cm^2

[1] H. Yukawa, *Proc. Phys.-Math. Soc. Japan*, **17**, 48 (1935).

[2] H. Hanstein, *Phys. Rev.*, **57**, 1045 (1940).

(*d*) the analysis of the proton–proton scattering shows that in the singlet state the interaction energy of two protons is, in very good approximation, the same as that between a proton and a neutron.

The nuclear forces must be dependent on the spins of the heavy particles, since the singlet state formed in the scattering of neutrons by protons is a virtual state and thus is higher than the triplet ground state of the deuteron by more than the binding energy of the deuteron.

The force needed—in addition to the Coulomb force—to explain the proton–proton scattering is practically the same as the force between proton and neutron. It would, therefore, be possible to explain nuclear forces with neutral mesons only. Since, however, the mesons found in cosmic rays are charged mesons, it seems more satisfactory to include charged mesons in the theory of nuclear forces. The assumption of charged mesons only, without neutral mesons, leads to difficulties in that, in first approximation, it does not give rise to forces between like particles.

Another difference between electromagnetic and meson-field theory is found in the fact that for charged particles at rest the term in the force quadratic in their charge gives the force exactly, namely the Coulomb force. For charged mesons the term quadratic in the coupling constant is only an approximation, even if the heavy particles are at rest. This corresponds to the fact that photons are not scattered by particles held at rest, while mesons are scattered by fixed heavy particles, since spin and charge do not stay "at rest."

The difficulty of electromagnetic mass appears in the meson theory even for fixed particles as the difficulty of inertia of spin and charge.

Various Types of Meson Fields

The heavy particles are characterized by wave functions ψ_N and ψ_P where the indices N, P indicate the two charge states, neutron and proton, of the nucleon (heavy particle). The operators of charge creation τ_+ and charge annihilation τ_- act on ψ as follows:

$$\begin{aligned} \tau_+\psi_N &= \psi_P & \qquad \tau_-\psi_P &= \psi_N \\ \tau_+\psi_P &= 0 & \qquad \tau_-\psi_N &= 0 \end{aligned}$$

where ψ_P and ψ_N are understood to mean

$$\psi_P = \binom{\psi}{0} \quad \text{and} \quad \psi_N = \binom{0}{\psi}$$

In this system the operators τ_+ and τ_- are, therefore, represented by the matrices

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the properties

$$\begin{aligned} \tau_+^2 &= \tau_-^2 = 0 \\ \tau_-\tau_+ + \tau_+\tau_- &= 1 \end{aligned}$$

We further introduce the operators

$$\begin{aligned} \tau_1 &= \tau_+ + \tau_- \\ \tau_2 &= -i(\tau_+ - \tau_-) \\ \tau_3 &= \tau_+\tau_- - \tau_-\tau_+ \end{aligned}$$

τ_1, τ_2, τ_3 have the same properties as the spin operators for spin $\frac{1}{2}$ and are called the "isotopic spin."

The meson field is represented by the operators of creation φ^* and annihilation φ of a meson, so that the operator $\tau_-\varphi^*$, for instance, changes a proton into a neutron and creates a meson of positive charge. Instead of φ and φ^* we shall sometimes use φ_1 and φ_2, defined by

$$\varphi = (\varphi_1 - i\varphi_2)/\sqrt{2} \qquad \varphi^* = (\varphi_1 + i\varphi_2)/\sqrt{2}$$

Scalar Charged Meson Field

The field energy of the meson field alone is written[3] as

$$H_0 = \tfrac{1}{2}\sum_{\alpha}\int [\pi_\alpha{}^2 + (\nabla\varphi_\alpha)^2 + \mu^2\varphi_\alpha{}^2]\,dV$$

and the interaction energy

$$H_{\text{int}} = \sqrt{4\pi}\, g \sum_{A,\alpha} \tau_\alpha^A \varphi_\alpha(Z^A)$$

where A (and in subsequent equations, B) denotes the heavy particles and Z^A the position of the heavy particles. From the Hamiltonian $H = H_0 + H_{\text{int}}$, the equations of motion are derived as follows:

$$\delta H/\delta\varphi_\alpha = -\dot{\pi}_\alpha = -\nabla^2\varphi_\alpha + \mu^2\varphi_\alpha + \sqrt{4\pi}\, g \sum_A \tau_\alpha^A \delta(X - Z^A)$$

$$\delta H/\delta\pi_\alpha = \dot{\varphi}_\alpha = \pi_\alpha$$

where δ indicates variational differentiation. Combining the equations of motion yields the Kirchhoff equation of the scalar meson field

$$-\nabla^2\varphi_\alpha + \mu^2\varphi_\alpha + \frac{\partial^2\varphi_\alpha}{\partial t^2} = -\sqrt{4\pi}\, g \sum_A \tau_\alpha^A \delta(X - Z^A)$$

Neglecting the time variation of the operators τ_α^A, one obtains a static solution for φ_α:

[3] All equations are written with natural units, i.e., $\hbar$ and $c = 1$.

$$(\varphi_\alpha)_{\text{stat}} = -\frac{g}{\sqrt{4\pi}} \sum_A \tau_\alpha^A \frac{e^{-\mu r}}{r}$$

where $r = |\mathbf{X} - \mathbf{Z}^A|$.

Substituting this solution into H_{int} we have

$$(H_{\text{int}})_{\text{stat}} = -g^2 \sum_{AB} \sum_\alpha \tau_\alpha^A \tau_\alpha^B e^{-\mu r_{AB}}/r_{AB}$$

with $r_{AB} = |\mathbf{Z}^A - \mathbf{Z}^B|$, so that the potential energy between two nucleons is

$$V_{AB} = -g^2 \sum_\alpha (\tau_\alpha^A \tau_\alpha^B)\, e^{-\mu r_{AB}}/r_{AB}.$$

If only charged mesons are assumed, α has the values 1 and 2, and we have

$$\sum_{\alpha=1,2} \tau_\alpha^A \tau_\alpha^B = 2(\tau_+^A \tau_-^B + \tau_-^A \tau_+^B)$$

with τ_1 and τ_2 as previously defined.

It is therefore

$$\sum_{\alpha=1,2} \tau_\alpha^A \tau_\alpha^B = \begin{cases} 0 \text{ for equal nucleons} \\ 2 \times \text{exchange of charge operator for} \\ \text{unequal nucleons} \end{cases}$$

[for instance:

$$2(\tau_+^A \tau_-^B + \tau_-^A \tau_+^B)\psi_P^A \psi_P^B = 0 \text{ because } \tau_+ \psi_P = 0$$
$$2(\tau_+^A \tau_-^B + \tau_-^A \tau_+^B)\psi_P^A \psi_N^B = 2\psi_N^A \psi_P^B]$$

This form of the theory, therefore, does not explain forces between like particles, but holds only in terms of a g^2 approximation. There are forces of order g^4 and of higher orders between like particles and, in strong coupling theories, these forces may be as strong as the g^2 forces.

Pseudoscalar Symmetric Theory

As mentioned above, the spin dependence of nuclear forces is necessary to explain the energy difference between the triplet and singlet state of the deuteron. The spin dependence can be achieved by introducing a pseudoscalar meson field, and forces between like nucleons are obtained by introducing neutral mesons. The unperturbed Hamiltonian of the meson field is

$$H_0 = \tfrac{1}{2}\sum_{\alpha} \int dV\{\pi_\alpha^{\,2} + (\nabla\varphi_\alpha)^2 + \mu_0^{\,2}\varphi_\alpha^{\,2}\}$$

and the interaction part is assumed to be

$$H_{\text{int}} = -\sqrt{4\pi}\sum_{A\alpha} \tau_\alpha f_0 \cdot (\boldsymbol{\sigma}_A \cdot \nabla)\varphi_\alpha(Z_A)$$

where f_0 is the coupling constant with the dimension of length, and the $\boldsymbol{\sigma}_A$ are the spin operators operating on the wave functions describing the nucleons. α runs from 1 to 3 for charged and neutral mesons. This form of H_{int} has been chosen because it is the simplest way of introducing spin dependent forces. The equations of motion are obtained in the same way as before and, combined, yield

$$\left(-\nabla^2 + \mu_0^{\,2} + \frac{\partial^2}{\partial t^2}\right)\varphi_\alpha = -\tau_\alpha \sum_A \sqrt{4\pi} f_0 \cdot (\boldsymbol{\sigma}_A \cdot \nabla)\delta(X - Z_A)$$

The potential energy for the interaction of two nucleons is obtained in the same way as before:

$$V_{AB} = f_0^{\,2} T_{AB}(\boldsymbol{\sigma}_A \cdot \nabla)(\boldsymbol{\sigma}_B \cdot \nabla)\, e^{-\mu_0 r_{AB}}/r_{AB}$$

where

$$T_{AB} = \sum_{\alpha=1,2,3} \tau_\alpha^A \tau_\alpha^B = \begin{cases} -3 \text{ for charge-antisymmetric states} \\ +1 \text{ for charge-symmetric states} \end{cases}$$

An example of a charge-antisymmetric state is the ground state of the deuteron, which is symmetric in orbit and spin and, therefore, antisymmetric in the charge, while the singlet state of the deuteron is spin-antisymmetric and orbit-symmetric, and, therefore, charge-symmetric.

Carrying out the differentiations we have

$$(\boldsymbol{\sigma}_A \cdot \boldsymbol{\nabla})(\boldsymbol{\sigma}_B \cdot \boldsymbol{\nabla}) \frac{e^{-\mu_0 r_{AB}}}{r_{AB}} = \frac{1}{3} S_{AB} \left(\frac{3}{r^3} + \frac{3\mu_0}{r^2} + \frac{\mu_0{}^2}{r} \right) e^{-\mu_0 r} + \frac{1}{3} \sum\nolimits_{AB} \cdot \frac{\mu_0{}^2}{r} e^{-\mu_0 r}$$

where

$$S_{AB} = 3(\boldsymbol{\sigma}_A \cdot \mathbf{n})(\boldsymbol{\sigma}_B \cdot \mathbf{n}) - (\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B), \quad \mathbf{n} = (\mathbf{X}_A - \mathbf{X}_B)/r_{AB}$$

and

$$\sum\nolimits_{AB} = (\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B)$$

The existence of the tensor force S_{AB} explains the existence and sign of the quadrupole moment. The high singularity of V_{AB} at $r_{AB} = 0$, however, makes the solution of the Schrödinger equation impossible. Bethe[4] has suggested cutting off at a small value of r_{AB}; in this way, one can actually explain the experimental data *a*, *b*, and *c*. It should, however, be noted that one then has three constants available, namely, μ_0, f_0, and the cutting-off radius.

Scattering of high-energy neutrons on protons is not in agreement with the pseudoscalar symmetrical theory. Hulthén[5] shows that a mixture between scalar neutral meson field and pseudoscalar charged meson field

[4] H. A. Bethe, *Phys. Rev.*, **57**, 260, 390 (1940).

[5] L. Hulthén, *Kgl. Fysiograf. Sällskap. Lund Förh.*, **14**, No. 2 (1944).

avoids this discrepancy. His theory has the further consequence that there is no tensor force between like nucleons, because the pseudoscalar charged mesons do not contribute to the interaction of like nucleons in the g^2 approximation. This is no objection to his theory since there is no experimental evidence for such a tensor force in the low-energy scattering experiments, which are the only ones available to the present time. Even if there were a tensor force between like nucleons, it could not be effective in these low-energy experiments since low energy implies orbital symmetry, and the likeness of the particles implies charge symmetry, so that the scattering must be described by a wave function antisymmetric in the spins. The tensor force S_{AB} vanishes when it is applied to a spin-antisymmetric wave function. Experiments on high-energy scattering of like particles would therefore be required to test Hulthén's theory.[6]

Instead of cutting off, Møller and Rosenfeld used a mixture of vector and pseudoscalar fields, a scheme which was improved by Schwinger.

Vector-Field Theory[7]

The Hamiltonian for the unperturbed field is

$$H_0 = \frac{1}{2} \sum_\alpha \int dV \left\{ \Pi_\alpha{}^2 + \frac{1}{\mu_1{}^2} (\nabla \cdot \Pi_\alpha)^2 + (\mathrm{curl}\, \psi_\alpha)^2 + \mu_1{}^2 \psi_\alpha{}^2 \right\}$$

The interaction part of the Hamiltonian is assumed to be

[6] The saturation question in connection with this hypothesis is discussed by L. Hulthén, *Phys. Rev.*, **67**, 193 (1945).

[7] N. Kemmer, *Proc. Roy. Soc. London*, **A166**, 127 (1938). H. J. Bhabha, *ibid.*, **A166**, 501 (1938).

$$H_{\text{int}} = -\sqrt{4\pi} \sum_{A,\alpha} \tau_{\alpha A} [f_1 \boldsymbol{\sigma}_A \,\text{curl}) \psi_\alpha(Z_A) + (g_1/\mu_1) \,\text{div}\, \Pi_\alpha(Z_A)]$$

where f_1 and g_1 are the coupling constants. The interaction energy between two nucleons is then found to be

$$V_{AB} = T_{AB}\{-(\boldsymbol{\sigma}_A \cdot \nabla)(\boldsymbol{\sigma}_B \cdot \nabla) f_1^2(e^{-\mu_1 r_{AB}}/r_{AB}) + (\Sigma_{AB} f_1^2 + g_1^2)\mu_1^2 \, e^{-\mu_1 r_{AB}}/r_{AB}\}$$

The tensor force is the same as in the pseudoscalar theory, except for the opposite sign. This makes the sign of the quadrupole moment wrong; the vector-field theory alone is therefore ruled out.

Møller and Rosenfeld mix the pseudoscalar and vector fields, assuming $f_1 = f_0$ and $\mu_1 = \mu_0$. In their theory, therefore, the tensor forces are completely cancelled, and the quadrupole moment vanishes.

Schwinger retains $f_1 = f_0$, but takes $\mu_1 > \mu_0$, so as to eliminate only the singularities higher than r_{AB}^{-1} and to obtain the correct sign of the quadrupole moment. The interaction energy is then

$$V_{AB} = \frac{1}{3} T_{AB} \left\{ S_{AB} f_0^2 \left[\left(\frac{3}{r^3} + \frac{3\mu_0}{r^2} + \frac{\mu_0^2}{r} \right) e^{-\mu_0 r} - \left(\frac{3}{r^3} + \frac{3\mu_1}{r^2} + \frac{\mu_1^2}{r} \right) e^{-\mu_1 r} \right] + \Sigma_{AB} f_0^2 \left[\mu_0^2 \frac{e^{-\mu_0 r}}{r} + 2\mu_1^2 \frac{e^{-\mu_1 r}}{r} \right] + 3 g_1^2 \mu_1^2 \frac{e^{-\mu_1 r}}{r} \right\}$$

Jauch and Hu[8] have solved the Schrödinger equation for this potential energy and find that under the assumption $g_1 = 0$ the binding energy of the deuteron and proton–neutron scattering cross section are obtained correctly by using $\mu_1/\mu_0 = 1.6$ and $f_0\mu_0 = 0.226$.

[8] J. M. Jauch and N. Hu, *Phys. Rev.*, 65, 289 (1944).

With this choice of constants they obtain for the quadrupole moment of the deuteron, however, the value 0.926×10^{-27} cm^2, instead of the experimental value of 2.73×10^{-27} cm^2. If one assumes g_1 to be different from zero this discrepancy becomes even worse.

Hulthén[9] has attempted to compute the quadrupole moment of the deuteron for the M–R mixture by taking into account the first order relativistic effect in the interaction energy between mesons and nucleons. The relativistic corrections thus obtained are large but cannot be trusted because the approximation used is equivalent to an expansion in terms of $(1/M)\nabla$, where M is the nucleon mass, and therefore introduces higher and higher singularities in the interaction potential between two nucleons.

The use of the rigorous relativistic form of the interaction between nucleons and the meson field and the taking into account of the recoil energy of the nucleons in the intermediate states does not improve the situation because even in all "mixed" theories singularities of the type r_{AB}^{-2} and r_{AB}^{-3} reappear.[10] The necessity of "cutting off" remains then even in the mixed theory, so that nothing is gained by the mixing.

[9] L. Hulthén, *Arkiv. Mat. Astron. Fysik*, **A29**, No. 33 (1943).
[10] N. Hu, *Phys. Rev.*, **67**, 339 (1945).

CHAPTER II

The nuclear forces considered in the first lecture were based on perturbation calculations, taking into account only terms quadratic in the coupling constant. But the discussion of higher-order terms leads to more than just mathematical difficulties. For point-source nucleons the theory is not convergent, as each step in the successive approximations results in higher and higher singularities and infinities. On the other hand the assumption of a finite source which makes the theory convergent is in contradiction with the relativistic invariance of the theory, unless particular tricks of subtraction are applied.

The concept of a finite source can be brought into field theories in two different ways. In electromagnetic theory, for instance, a radius "a" of the charged particle can be introduced as a minimum radius inside of which the Coulomb field does not hold. The electromagnetic mass can be calculated from this radius a. As an alternative, one could assume the Coulomb field to stay formally valid everywhere, but replace infinite terms in the interaction energy by newly chosen mechanical constants.

In meson theory a mechanical constant $1/a$ of dimension length^{-1} can be chosen; it is found to determine the "spin inertia," that is, the reaction of the eigenfield of the nucleon to the motion of its spin. Unlike the electron mass, the spin inertia is not experimentally known. But we shall see that it cannot be zero, because this assumption would lead to wrong values for the magnetic moments of the nucleons. The assumption of large spin inertia leads to the so-called "strong coupling" case and results in the existence of excited states (isobars) of the nucleon with higher values of the spin. Effects involving the spin, similar to those mentioned here, result from the interaction of charged meson fields with the "isotopic spin" of the nucleons. The same constant a^{-1} will, in fact, determine also a "charge inertia" and in these theories excited states of the nucleon with higher values of the charge will exist.

Experimentally, the existence of such isobars should affect experiments on high-energy 30–40 Mev scattering of nucleons on nucleons. If these stable isobars exist they should be created in scattering processes, where the incident neutrons have an energy larger than, or equal to, twice the excitation energy of the isobars.

Theory of Extended Source

In this theory the heavy particle is assumed to be at rest. For the meson field, which of course must be treated relativistically, the neutral pseudoscalar case is chosen for simplicity. The Hamiltonian for the meson field in interaction with one nucleon is

$$H = \tfrac{1}{2}\int \left\{\pi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2\right\} d^3x + \sqrt{4\pi}\, f \int U(\mathbf{x})(\boldsymbol{\sigma}\cdot\nabla)\varphi\, d^3x$$

where d^3x denotes the volume element in x-space, and the integration extends over all space. For $U(\mathbf{x})$ we only assume now $\int U(\mathbf{x})d^3x = 1$. For a point source, $U(\mathbf{x})$ would be a δ-function.

We shall now perform a transformation to momentum space by means of the following equations:

$$U(\mathbf{x}) = (2\pi)^{-3}\int v(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d^3k$$

with

$$v(\mathbf{k}) = \int U(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x$$

From the normalization of $U(\mathbf{x})$ follows $v(0) = 1$. The wave functions are transformed as follows:

$$\varphi(\mathbf{x}) = (2\pi)^{-3/2}\int q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d^3k$$

with

$$q(\mathbf{k}) = (2\pi)^{-3/2}\int \varphi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x$$

and

$$\pi(\mathbf{x}) = (2\pi)^{-3/2}\int p(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}d^3k$$

with

$$p(\mathbf{k}) = (2\pi)^{-3/2}\int \pi(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}d^3x$$

We further define

$$\begin{aligned} \tilde{q}(\mathbf{k}) &= q(\mathbf{k})v(-\mathbf{k}) \\ \tilde{p}(\mathbf{k}) &= p(\mathbf{k})/v(-\mathbf{k}) \\ G(\mathbf{k}) &= v(\mathbf{k})\, v(-\mathbf{k}) \end{aligned}$$

In its original meaning, U, describing the extension of a nucleon in space, should be real. Then $v(-\mathbf{k})$ must be equal to $v^*(\mathbf{k})$, and $G(\mathbf{k})$ is positive. We shall, in the following, generalize by assuming G real but not necessarily positive. This generalization is possible in momentum space, and as long as one does not consider quantities like the spatial densities of mesons. While with this generalization U is not necessarily real, H

can still be kept real, as long as $\tilde{q}(-\mathbf{k}) = \tilde{q}^*(\mathbf{k})$. We find for H:

$$H = \frac{1}{2}\int d^3x(2\pi)^{-3}\Big\{\int\int p(\mathbf{k})\, p(\mathbf{k}')\, e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} d^3k d^3k' +$$

$$\int\int [-(\mathbf{k}\cdot\mathbf{k}') + \mu^2]\, q(\mathbf{k})q(\mathbf{k}')e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} d^3k d^3k'\Big\} + \sqrt{4\pi} f \times$$

$$\int d^3x(2\pi)^{-3}\int v(\mathbf{k}')e^{i\mathbf{k}'\mathbf{x}} d^3k' \int (\boldsymbol{\sigma}\cdot i\mathbf{k}')(2\pi)^{-3/2} q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} d^3k$$

$$= \frac{1}{2}\int p(\mathbf{k})p(-\mathbf{k})d^3k + \int (\mu^2 + k^2)q(\mathbf{k})q(-\mathbf{k})d^3k + \frac{if}{\sqrt{2}\cdot\pi}\int (\boldsymbol{\sigma}\cdot\mathbf{k})\, q(\mathbf{k})\, v(-\mathbf{k})d^3k$$

$$= \frac{1}{2}\int \Big\{G(\mathbf{k})\tilde{p}(\mathbf{k})\tilde{p}(-\mathbf{k}) + \frac{k_0^2}{G(\mathbf{k})}\tilde{q}(\mathbf{k})\tilde{q}(-\mathbf{k})\Big\} d^3k + \frac{if}{\pi\sqrt{2}}\int (\boldsymbol{\sigma}\cdot\mathbf{k})\tilde{q}(\mathbf{k})\, d^3k$$

with $k_0^2 = k^2 + \mu^2$.

This Hamiltonian yields the following equations of motion:

$$-\frac{\delta H}{\delta\tilde{q}} = \dot{\tilde{p}}(\mathbf{k}) = -\frac{k_0^2}{G(\mathbf{k})}\tilde{q}(-\mathbf{k}) - \frac{if}{\sqrt{2}\pi}(\boldsymbol{\sigma}\cdot\mathbf{k})$$

or

$$\dot{\tilde{p}}(-\mathbf{k}) = -\frac{k_0^2}{G(\mathbf{k})}\tilde{q}(\mathbf{k}) + \frac{if}{\sqrt{2}\pi}(\boldsymbol{\sigma}\cdot\mathbf{k})$$

and

$$\dot{\tilde{q}}(\mathbf{k}) = \frac{\delta H}{\delta\tilde{p}(\mathbf{k})} = G(\mathbf{k})\tilde{p}(-\mathbf{k})$$

Combining these two equations yields

$$\frac{\partial^2\tilde{q}(\mathbf{k})}{\partial t^2} + k_0^2\tilde{q}(\mathbf{k}) = \frac{if}{\pi\sqrt{2}}G(\mathbf{k})(\boldsymbol{\sigma}\cdot\mathbf{k}) \qquad (1)$$