C. CARATHÉODORY

CALCULUS OF VARIATIONS AND PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Translated by Robert B. Dean
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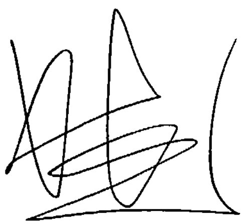

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PREFACE TO THE SECOND ENGLISH EDITION

For compelling reasons, the first English-language edition of Carathéodory's famous book on the calculus of variations was published in two separate (but consecutively paged) volumes. In this Second Edition, the two volumes have been combined into one. This has entailed some minor rearrangement of the Prefaces and Bibliographies and the combining of the two indexes into a single index. The advantages to the reader of having the book in one volume is obvious.

In this Second Edition, errata have, of course, been corrected. In addition, revisions have been made to enhance the clarity of the English text.

A. G.

PREFACE TO THE FIRST ENGLISH EDITION

Part I

This English version of C. Carathéodory's masterpiece results from both the need and desire to make the work accessible to a wider circle of readers than was possible in the German language edition. The original text was divided into two parts, the first part was entitled" Partial Differential Equations of the First Order" and the second, "Calculus of Variations."

In 1956 a German edition of the first part was published under the guidance and editorship of Professor E. Hölder. In this 1956 edition the few errors which appeared in the first edition, both typographical and mathematical, were corrected; in addition Professor Hölder included a supplement of his own and brought the literature up to date. All of this has been retained in the present publication, with very minor exceptions.

The influence of Carathéodory's ideas, which are manifest in his research publications, monographs and treatises, cannot be fully assessed since they continue to be a fruitful source in mathematical analysis and mathematical physics. It is therefore not conjecture to state that this influence will continue in the future. The geometrical content which underlies the present version and also Part II should provide the reader with the necessary equipment to understand better the theory and applications of differential forms, both in mathematics and the physical sciences. A conscious effort was maintained to preserve the stylistic flavor of Carathéodory's writing, and although this is a difficult task in any translation, we hope that it has been achieved here.

We wish to thank William C. Schulz for reviewing the manuscript in detail for clarity of exposition and for his comments on other possible flaws in the early drafts of the translation.

Julius J. Brandstatter
Stanford Research Institute

PREFACE TO THE FIRST ENGLISH EDITION

Part II

The author's preface describes both his intent and the contents of the present volume, hence there is no need to embellish his remarks. Since the appearance of the original German edition an enormous amount of literature on the calculus of variations and related fields has been published. To see this, one need only consult the American Mathematical Reviews and announcements of dissertations on the subject. During the past two decades the physical applications and computational algorithms based on the concepts of the variational calculus have grown to voluminous proportions, as evidenced by the journal articles throughout the world. Carathéodory's work provided the stimulus for many of these research papers and monographs. In order to make the reader aware of this newer work, a list of significant references has been added at the end of Carathéodory's original guide to the literature. These references exemplify the scope and directions that studies in the calculus of variations have taken in modern times. It is hoped that the present translation will continue to open new vistas to the English-reading audience.

JULIUS J. BRANDSTATTER

PREFACE TO THE FIRST GERMAN EDITION

Almost one hundred years ago Jacobi[1] discovered that the differential equations which occur in the calculus of variations, and partial differential equations of the first order, are connected with each other, and in particular that to each such partial differential equation there correspond variational problems. For the more special problems of geometrical optics this interrelation between the calculus of variations and partial differential equations had already been observed a decade earlier by W. R. Hamilton, whose work, moreover, influenced Jacobi. But Hamilton essentially had done no more than answer the age-old problem that had been raised through the dual founding of geometrical optics by Fermat's and Huygens' principles.

Although the statement of the problem itself and the conclusions resulting from it are now quite old, the consequences which follow from it have been only slightly comprehended until now. Among these, one must mention in the first place several marvelous works of Beltrami, who investigated the relations between Gauss's theory of surfaces and the results of Jacobi.[2] On the other hand, in the cultivation of the calculus of variations, neither Jacobi nor his students, nor the many other prominent men who so brilliantly represented and advanced this discipline during the nineteenth century, thought in any way of the relationship that connects the calculus of variations with the theory of partial differential equations. This is even more striking since most of these famous mathematicians were specifically concerned with partial differential equations of the first order. Indeed it appears that the original observation of Jacobi was regarded—even by himself—not as the fundamental fact it really is, but rather as a formal coincidence.

Not until the turn of the century did the cloud lift a little when Hilbert around 1900 introduced his "independent integral" into the Weierstrass theory of the calculus of variations. But it is certainly only by chance that Hadamard, according to a remark in his book,[3] did not pursue further the relationship concerning us here, which he saw extraordinarily clearly.

For many years it was my wish to put this complex of ideas, which

[1] C.G.J. Jacobi, *Jur Theorie der Variations-Rechnung und der Differential-Gleichungen* (Schreiben an Herrn Encke, Secretar der math.- phys. KL. der Akad. d. Wiss. zu Berlin, vom 29. Nov. 1863). *Ges. Werke* Bd. V, pp. 41-55.

[2] E. Beltrami, *Opere Matematiche* (Milano, Hoepli 1902) T. I, pass. cf. in particular, pp. 115 and 366.

[3] J. Hadamard, *Leçons sur le Calcul des Variations*, p. 151.

remained unnoticed for so long, into the proper perspective. For this purpose it was necessary to examine the entire theory anew, and it is not strange that the preparation for this volume required much time. The book consists of two parts. In the first part I have made an attempt to simplify the presentation of the theory of partial differential equations of the first order so that its study will require little time and also be accessible to the average student of mathematics. In this presentation the methods of S. Lie had to remain unconsidered in many ways; however, since there are good modern books (for example that of Engel and Faber) which emphasize the viewpoint of Lie, this sin of omission is of lesser importance than would appear at first glance.

The second part, which contains the calculus of variations, can also be read independently if one refers back to earlier sections in Part I. Moreover, there is no pretense of completeness: only those chapters in which the fundamentals of the calculus of variations are discussed are treated in all of the necessary details; beyond that the theory is merely pursued up to those points from which independent study can start. In many sections of the book the problems raised only serve as examples by which one can test the power of the general methods and learn their wide scope of applicability; for an entire series of questions, such as discontinuous solutions, transformation theory of variational problems, integral invariants, Finsler spaces, as well as the treatment of problems which depend on multiple integrals, that were to have been included in this volume according to the original plan, in agreement with the publisher were omitted so that, the text could be priced to be accessible to those scientific circles to whom the material is applicable. Since however, Part I of the book was not curtailed. I hope to have furnished the reader with all the building blocks to enable him to enlarge the existing structure according to his needs. The "guide to the use of the literature," which has been compiled at the end of the book, should serve the same purposes.

I have never lost sight of the fact that the calculus of variations, as it is presented in Part II, should be above all a servant of mechanics. Therefore, I have prepared in particular all questions to be treated from the outset for multidimensional spaces. I have especially emphasized some of the closer connections between both disciplines.

The purpose of the whole work will have been attained if it is capable of convincing the expert in this branch of mathematics that there exist today in the calculus of variations three basic approaches: first, the variational calculus of Lagrange, which now forms a part of tensor calculus, second, the theory of Tonelli, in which the more subtle relations of the minimum problem to set theory are developed; finally, the approach set forth in this work, which is oriented to the theory of differential equations, to differential geometry, and to the physical applications which first attained prominence through Euler in his *Methodus inveniendi lineas curvas......* I hope to have demonstrated that the Weierstrass theory of the calculus variations also belongs to the latter approach.

I should like to express my thanks to W. Damköhler, A. Duschek, N. Kritikos, and A. Rosenthal for their aid in corrections of the manuscript and for many improvements of the text that have resulted from their collaboration.

The more complicated figures and also Figures 6 and 7 in Chapter 14 as well as most of the figures of Chapter 16 were calculated by Dr. J. Meixner and neatly drawn by Mr. H. Steigerwald.

Finally my thanks are extended to the publishers who not only agreed to all my wishes, but who also prepared the rather difficult typesetting and additional figures with their usual exemplary perfection.

Munich, April 1935.

C. Carathéodory

BIOGRAPHICAL NOTE

Constantin Carathéodory was born in Berlin on September 13, 1873, but ıe grew up in Brussels where his father was the Turkish ambassador to Belgium from 1875 to 1900. He came from a respected Greek family which ıad lived in Constantinople since the beginning of the nineteenth century ınd whose members had held many important diplomatic and governmental ›ositions.

Carathéodory's preliminary education ended with graduation from a *Gymnasium*; he then entered the *Ecole Militaire de Belgique*, where four ·ears later he finished training as an engineering officer. After additional echnical studies in London and Paris he went to Egypt to work as an engiıeer along the Nile.

In 1900 he decided to go to Germany to devote himself exclusively to ıathematics. He studied four years in Berlin and Göttingen as a student f Schwarz and Hilbert and graduated from Göttingen in 1904. His academic ·ork as a lecturer at Göttingen and Bonn and later his duties at the technical *Iochschulen* in Hanover and Breslau met with such success that in 1913 he ucceeded Professor Felix Klein at the University of Göttingen. Later, in 918, he went to the University of Berlin.

After only two years in Berlin he accepted the request of the Greek overnment to take over the founding and organization of the newly planned reek univerity in Smyrna. His accomplishments in establishing the instiıte and in obtaining well-known Greek and foreign professors were, however, estroyed when Turkish troops occupied and burned the city.

For the next two years he served at the university and technical *Hochschule* Athens and in 1924 finally accepted a position at the University of Munich here he remained until his death on February 2, 1950.

Constantin Carathéodory's success as a mathematician was greatly aided ' his gift for languages, a talent which he possessed to a high degree. He ıew both Greek and French as native tongues and mastered German to ch perfection that most of his publications in that language are considered be stylistic masterpieces. In addition to this he spoke English, Italian, d Turkish and read both classical Latin and Greek literature as an evening version. Mastery of so many languages enabled him to communicate freely d cordially with researchers of many nations on his extended foreign ıvels and greatly to enlarge his field of vision and various spheres of prossional activity.

R. D.

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PART I

PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

The parts of analysis we shall use in presenting the calculus of variations in Part II of this work form a sharply defined discipline, which itself may claim the interest of mathematicians. It concerns facts that were developed in the course of the nineteenth century independently and for various reasons, and yet when taken together form a single imposing structure.

This is not the only reason making it advantageous to separate the theories of partial differential equations, of importance in this Part I, from the trains of thought of the actual calculus of variations (Part II), and to study these theories in all the necessary detail; for we will often be preparing a firm basis from which the calculus of variations, which is related to the most diverse branches of mathematics and their physical applications, can grow in all directions. In this way the means will be accessible to the reader of this work to handle independently a number of questions from the calculus of variations which for various reasons we had to leave unconsidered.