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PREFACE

There are several ways of producing new functions under certain conditions when two functions f and T are given, and one of them is to compose them, *i.e.*, to evaluate the old function f at the new points $T(x)$ whenever the range of T is a subset of the domain of f. This new function, as is well known, is called the composite of f and T and is denoted by the symbol $f \circ T$. If f varies in a linear space of functions on the range of T with pointwise linear operations, then the mapping taking f into $f \circ T$ is a linear transformation. This correspondence is known as the composition transformation induced by the function T and we denote it by the symbol C_T . Under certain conditions on T and the linear space on which C_T acts it turns out that C_T is a nice and well behaved operator. Another way of producing a new function when two functions π and f are given is to multiply them whenever it makes sense. This gives rise to a linear transformation known as the multiplication transformation induced by π. The multiplication transformations and the composition transformations under suitable situations breed another class of transformations, known as the weighted composition transformations which we denote by $W_{\pi,T}$ and define as

$$W_{\pi,T}(f) = \pi \cdot f \circ T.$$

This monograph presents a study of these operators on different spaces of functions. So far as we think, the study of the composition operators was initiated with two goals in mind, namely, to have concrete examples of bounded operators on Hilbert spaces or Banach spaces of functions, and to attack the "Invariant Subspace Problem" of functional analysis from a different angle. These goals, as I think had their origin in the thinking of P.R. Halmos and were grasped by E.A. Nordgren who started a systematic study of these operators in 1968. During the last 25 years or so quite a bit has been achieved in respect of the first goal; many concrete examples of the composition operators on L^p–spaces, H^p–spaces or topological vector spaces of functions have been obtained giving a place to these operators in 1991 mathematics subject classification of Mathematical Reviews (47B38). It is interesting to note that in certain situations this class of operators possesses a distinct identity different from several known classes of operators, like the multiplication operators and the integral operators. No composition operator on ℓ^2 is compact, but every composition operator on ℓ^2 has a non-trivial invariant subspace. Similarly, the general invariant subspace problem of operators on a Hilbert space H is linked with the minimal invariant subspaces of an invertible composition operator C_T on $H^2(D)$ [257]. This has been a step forward in the direction of achieving the second goal and much more still is to be achieved.

Whenever a study is undertaken with some specific aims in mind, a lot of other things crop up as byproducts having connections with other branches of knowledge. This has happened with the study of the composition operators too. These operators are being used in statistical mechanics, distribution theory and topological dynamics besides their earlier applications in ergodic theory and classical mechanics.

If we look back at the works done on these operators, then it becomes evident that the most of the study has been concentrated on L^p–spaces, H^p–spaces or the locally convex spaces of continuous functions. An endeavour has been made in this monograph to present most of the results obtained so far. We have been a little prejudiced in presenting the materials in chapter II and chapter IV due to our special interest in the composition operators on L^p–spaces and on the weighted spaces of continuous functions. In order to contain the monograph within the limits some of the results are stated without proofs with appropriate references for the proofs. The background materials are not presented, and hence the readers are expected to have some knowledge of measure theory, analytic function theory and functional analysis to have a smooth sailing through the monograph. Besides being a reference book for researchers, this book may be used for a topic course to the advanced post-graduate students.

Chapter I starts with a broad and unified definition of the composition operator and the weighted composition operator and introduces the concrete spaces of functions on which the study of these operators is carried out in the subsequent chapters. A historical development of the theory of these operators is also presented in this chapter.

Chapter II deals with the composition operators on L^p–spaces. The composition operators on L^p–spaces are characterised and many examples are presented to illustrate the theory. Invertibility, compactness and different types of normality of these operators are studied in this chapter.

The composition operators and the weighted composition operators on functional Banach spaces are the subject matter of chapter III. H^p–spaces and ℓ^p–spaces are concrete examples of functional Banach spaces and results pertaining to the composition operators on these spaces are reported in this chapter. In chapter IV of this monograph we have studied the composition operators and the weighted composition operators on some locally convex spaces of continuous functions and cross sections. These spaces are broad enough to include many nice spaces which are used in analysis.

The composition operators and the weighted composition operators are employed in characterizations of isometries and homomorphisms on some spaces of functions. They have been utilised in the dynamical systems to study different types of motions. The ergodic theory and topological dynamics make use of the composition operators in developement of their theories. Some of these applications of these operators are presented in the last chapter of this book.

We have tried our best to collect and present many known results about these interesting operators, but still many results might not have found their place in this monograph. They can be found in the cited references which we have tried to keep as updated as possible. There will probably be some more references which are not included ; this is because we might not have seen them. We would like to apologise to those whose papers are not listed here. Most of the symbols used are standard ones, like H^p–spaces, L^p–spaces etc., for Hardy spaces, Lebesgue spaces etc., and they are introduced in chapter I. The number 2.4.5 indicates the fifth item of section 4 of chapter II and a reference such as [94] refers to the entry at No. 94 of the bibliography.

The work on this monograph began in the Fall of 1985 when I visited the University of Arkansas, U.S.A., as a Fulbright Faculty Fellow under Indo–U.S. fellowship programme. My several visits to the University of Massachusetts at Boston provided me opportunity to continue my work on this book. Thus five agencies have been involved in the support of the preparation of this monograph, they are : the University of Jammu, the University of Arkansas, the University of Massachusetts, the Council for International Exchange of Scholars and the University Grants Commission. I express my deep sense of thankfulness to each of them.

Normally, behind every work there is some one who gives inspiration, encouragement and motivation ; in this case there has been a man who is a mathematician, has been my teacher and is a trusted friend of mine. His name is V.L.N. Sarma. He introduced me to mathematics, taught me mathematics and inspired me to work in mathematics. I have no words to record my indebtedness and gratitude to Professor Sarma. I am thankful to many colleagues of mine and some of my Ph.D. students who contributed directly or indirectly towards the preparation of this book. They are : Dr. D.K. Gupta, Dr. Ashok Kumar, Dr. B.S. Komal, Dr. S.D. Sharma, Dr. T. Veluchamy, Dr. D. Chandra Kumar, Mr. Bhopinder Singh, Mr. Romesh Kumar. Dr. J. S. Manhas deserves a special thanks as he has collaborated with me in preparation of the last two chapters of the monograph.

Professor William Summers of the University of Arkansas worked as the faculty

associate during my visit under Indo–U.S. fellowship programme and collaborated with me in writing some research articles on composition operators. Professors Herbert Kamowitz and Dennis Wortman of the University of Massachusetts at Boston have read the manuscript of the monograph. They encouraged and inspired me to complete the work whenever I visited them in Boston. I intend to record my deep sense of gratitude and thankfulness to all these three mathematicians. I am indebted to my wife, Krishna whose patience, endurance and encouragement contributed a lot towards the completion of this work.

I am thankful to Professor L. Nachbin of the University of Rochester for inviting me to write this monograph for the series Mathematics Studies. Our thanks are due to the publisher for his co-operation in bringing out this book. The manuscript was typed by Mr. Ranjit Singh and Mr. Manjit Singh of our department and the Camera-ready manuscript was prepared by Mr. Dalip R. Mohun of M/s Shiroch Reprographics, Allahabad. We are thankful to each of them for their excellent job and sincere work pertaining to the manuscript. Errors and mistakes still remaining in the book would be due to our negligence.

May, 1993 R.K. Singh

CONTENTS

CHAPTER I

INTRODUCTION

1.1 DEFINITIONS AND HISTORICAL BACKGROUND

Let X be a non-empty set and let F_x be a vector space over the field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) for every $x \in X$. Then the Cartesian product $\prod_{x \in X} F_x$ of the family $\{F_x : x \in X\}$ is a vector space under linear operations defined pointwise. Any element of $\prod_{x \in X} F_x$ is known as a cross-section over X, and the family $\{F_x : x \in X\}$ is known as a vector-fibration over X. By $L(X)$ we denote a topological vector space of the cross-sections over X. Let $T : X \to X$ be a mapping such that $f \circ T \in \prod_{x \in X} F_x$ whenever $f \in L(X)$. Then the correspondence $f \to f \circ T$ is a linear transformation from $L(X)$ to $\prod_{x \in X} F_x$ and it is called the composition transformation induced by T. This transformation is denoted by C_T. If π is a mapping defined on X such that $\pi.f \circ T \in \prod_{x \in X} F_x$ whenever $f \in L(X)$, then the correspondence $f \to \pi.f \circ T$ is a linear transformation from $L(X)$ to $\prod_{x \in X} F_x$. This transformation is called the weighted composition transformation induced by π and T, and we denote it by $W_{\pi,T}$. Our interest lies in the case in which C_T and $W_{\pi,T}$ take elements of $L(X)$ into elements of $L(X)$ and $C_T : L(X) \to L(X)$ and $W_{\pi,T} : L(X) \to L(X)$ are continuous. In this case we call C_T and $W_{\pi,T}$ the composition operator and the weighted composition operator on $L(X)$ induced by T and the pair (π, T) respectively. In case $F_x = \mathbb{K}$ for every $x \in X$, $\prod_{x \in X} F_x$ turns out to be the space of all scalar-valued functions on X and $L(X)$ is a topological vector space of complex-valued or real-valued functions. If $F_x = E$ for every $x \in X$, where E is a topological vector space over $\mathbb{K}$, then $\prod_{x \in X} F_x$ is the vector space of all vector-valued (E-valued) functions on X and $L(X)$ is a topological vector space of vector-valued (E-valued) functions. In this monograph an attempt is made to present a theory of these operators, (in particular of composition operators) on different topological vector spaces of functions based on the work done during the last two decades or so.

In order to have a mathematically fruitful and interesting theory of these operators it is desirable to have some structures (algebraic, topological, or a combination) on X and some conditions on the inducing mappings T and π. The following are three major situations occuring in the study of these operators :

(i) The underlying space X is a measure space and the inducing maps are measurable transformations,

(ii) The underlying space X is a region in $\mathbb{C}$ or $\mathbb{C}^n$ and the inducing maps are holomorphic functions,

(iii) The underlying space X is a topological space and the inducing maps are continuous functions.

In the first situation $L(X)$ is taken to be a topological vector space of measurable functions such as L^p–spaces; in the second case $L(X)$ is taken to be a topological vector space of analytic functions, like a Hardy space or a Bergman space or a Dirichlet space; and in the third case $L(X)$ is taken to be a topological vector space of continuous functions. There is quite a bit of overlap between these three cases and study becomes more interesting on the overlapping portion as it can be viewed from several angles. In the first situation in which $L(X)$ is an L^p–space the theory of the composition operators establishes a contact with ergodic theory, entropy theory, and classical mechanics; in the second case the theory touches differentiable dynamics, statistical mechanics, and the theory of distributions; and the third situation appears in topological dynamics, transformation groups, and in study of the continuous functions (see [136], [171], [237], [238], [270]).

So far as we know, the earliest appearance of a composition transformation was in 1871 in a paper of Schroeder [302], where it is asked to find a function f and a number α such that

$$(f \circ T)(z) = \alpha f(z)$$

for every z in an appropriate domain, whenever the function T is given. If z varies in the open unit disk and T is an analytic function, then a solution has been obtained by Koenigs [183] in 1884. In 1925 these operators were employed in Littlewood's subordination theory [221]. In the early 1930's the composition operators were used to study problems in mathematical physics and classical mechanics, especially worth mention is the work of Koopman ([199], [200]) on classical mechanics. In those days these operators were known as the substitution operators or translation operators. Banach used these operators to characterise isometries on Banach spaces of continuous functions [80]. In the 1940's and 1950's composition operators appeared in the work of Von Neumann and Halmos ([136], [140]) in the study of ergodic transformations. In 1966 Choksi [68] studied unitary composition operators and continued the study in later years.

The history of a systematic study of the composition operators is not that old. It was started in 1968 by Nordgren in his paper [253] though it had its appearance in 1966 in a paper of Ryff [297]. In his paper Nordgren studied composition operators on L^2 of the unit circle induced by inner functions. In 1969 Schwartz [303] and Ridge [278] wrote their Ph. D. theses on composition operators on H^p-spaces and composition operators on L^p-spaces, respectively. In 1972 Singh [324] completed his doctoral dissertation on composition operators under the supervision of Professor Nordgren. After this a group of mathematicians including Boyd [41], Caughran ([60], [61]), Cima ([73], [74]), Kamowitz ([160], [161]), Shapiro [309], Swanton ([384], [385]), and Wogen ([73], [74]) plunged into the study of composition operators on different function spaces. A series of lectures on composition operators by Nordgren at the Long Beach Conference on "Hilbert Space Operators" in 1977 ([254]) gave a further boost to the study of the composition operators and several more mathematicians such as Cowen ([83], [84]), Deddens [91], Iwanik [151], Roan [281], Whitley [399], etc., started their explorations of the properties of composition operators. In this way the 1970's had been a very fruitful decade so far as the study of composition operators on L^p-spaces and H^p-spaces is concerned. A group of research workers led by Singh had been engaged in studying composition operators at Jammu since 1973. The systematic study initiated in the 1970's has been continued and extended in several directions during the last decade. Worth mention are some new names such as Gupta [127], Komal [187], Kumar [202], Lambert [147], MacCluer ([228], [229], [230]), Mayer ([237], [238]), Sharma [311], Stanton [72], Summers [368], Takagi [387], Veluchamy [396], etc., joining the earlier group in exploration of the properties of the composition operators on different function spaces. In the later half of the last decade the study of these operators on spaces of continuous functions initiated by Kamowitz [164] picked up momentum. Feldman [109], Jamison and Rajagopalan [153], Singh and Summers [368], Singh and Manhas [352, 354], Takagi [388] made systematic studies of these operators on several spaces of continuous functions. Much has been known about this interesting, simple, but rich class of operators, but still there is much more to be explored.

This monograph aims at presenting a more or less consolidated and unified account of the systematic work done on the composition operators and related topics. Most of the results presented in this monograph are published in different mathematical journals; still there are many results which are either unpublished or in the process of publication. Particularly, Chapter IV contains many unpublished results which the authors have obtained recently.

Before proceeding to the next chapter, we would like to introduce the underlying spaces of functions on which the composition operators have been studied. We can divide these spaces into three broad categories :

(i) L^p–spaces.

(ii) Functional Banach spaces of functions.

(iii) Locally convex function spaces.

The next three sections of this chapter aim at introducing these spaces to the readers.

1.2 L^p-SPACES

Let $(X, \mathcal{S}, m)$ be a measure space and let p be a real number such that $1 \le p < \infty$. Let $\mathcal{L}^p(m)$ denote the set of all complex–valued measurable functions on X such that $|f|^p$ is m-integrable. Then $\mathcal{L}^p(m)$ is a complex linear space under the operations of pointwise addition and scalar multiplication. If $N^p(m)$ denotes the set of all null functions on X, then $N^p(m)$ is a subspace of $\mathcal{L}^p(m)$. Let $L^p(m)$ denote the quotient space $\mathcal{L}^p(m)/N^p(m)$. An element in $L^p(m)$ is a coset of the type $f+N^p(m)$, where f belongs to $\mathcal{L}^p(m)$. The coset $f+N^p(m)$ is denoted as $[f]$. Thus two functions g and h of $\mathcal{L}^p(m)$ belong to the same coset if and only if $g = h$ almost everywhere. On $L^p(m)$ we define a norm as

$$\|[f]\|_p = \left(\int |f|^p \, dm\right)^{\frac{1}{p}}.$$

Using the Minkowski inequality it can be shown that $L^p(m)$ is a normed linear space under the above norm. Under this norm $L^p(m)$ is complete. Thus $L^p(m)$ is a Banach space. The conjugate space of $L^p(m)$ is $L^q(m)$, where p and q are conjugate indices. For $p = 2$, $L^p(m)$ is a Hilbert space under the inner product defined as

$$\langle [f],\ [g] \rangle = \int f\bar{g}\ dm.$$

If X has a non-empty subset of measure zero, then it is evident that the elements of $L^p(m)$ are not functions on X but they are equivalence classes of functions on X. Two elements of $\mathcal{L}^p(m)$ are equivalent if they agree almost everywhere. Under this identification we regard $L^p(m)$ as a Banach space of functions. We shall take f as an element of $L^p(m)$ instead of taking $[f]$ as an element of $L^p(m)$. This function f represents all those functions of $\mathcal{L}^p(m)$ which are equal to f *a. e.*

If $X = \mathbb{N}$, the set of natural numbers, $\mathcal{S} = P(\mathbb{N})$, the power set of $\mathbb{N}$ and m the counting measure, then we denote the corresponding $L^p(m)$ by ℓ^p. This ℓ^p is the classical sequence space. A sequence $\{\alpha_n\}$ of complex numbers belongs to ℓ^p if $\sum_{n=1}^{\infty} |\alpha_n|^p < \infty$. The space ℓ^2 is the classical example of a Hilbert space given by Hilbert himself. If $w = \{w_n\}$ is a sequence of non-negative real numbers, and if the measure m on $P(\mathbb{N})$ is defined as

$$m(S) = \sum_{n \in S} w_n ,$$

then the corresponding $L^p(m)$ is denoted by $\ell^p(w)$. This is called the weighted sequence space with w as the sequence of weights.

A complex valued measurable function f on X is said to be essentially bounded if there exists an $M > 0$ such that the measure of the set

$$\{x \ :\ x \in X \text{ and } |f(x)| > M\}$$

is zero. By $\|f\|_\infty$ we denote the smallest such M, which is called the essential supremum of f. Let $\mathcal{L}^\infty(m)$ denote the set of all essentially bounded functions on X. Then $\mathcal{L}^\infty(m)$ is a linear space. By $L^\infty(m)$ we denote the quotient space $\mathcal{L}^\infty(m)/N^\infty$, where N^∞ is the subspace of null functions. With the essential supremum norm $L^\infty(m)$ becomes a Banach space. The symbol ℓ^∞ stands for the Banach space of all bounded sequences of complex numbers.

A measurable space $(X, \mathcal{B})$ is said to be a standard Borel space if X is a Borel subset of a complete separable metric space and $\mathcal{B}$ is the σ–algebra of Borel sets. Many measure theoretic pathologies in the study of composition operators are avoided if the underlying space is a standard Borel space. Most of the nice and useful examples of measurable spaces are standard Borel spaces.

1.3 FUNCTIONAL BANACH SPACES OF FUNCTIONS

Let X be a non-empty set and let $H(X)$ be a Banach space of complex–valued functions on X with pointwise addition and scalar multiplication. Let $x \in X$. Let δ_x be the mapping on $H(X)$ taking f into $f(x)$. Then it is obvious that δ_x is a linear functional on $H(X)$; it is called the evaluation functional induced by x. The space $H(X)$ is said to be a functional Banach space if each evaluation functional δ_x is continuous *i.e.*, if $\delta_x \in H^*(X)$ for every $x \in X$, where $H^*(X)$ denotes the dual space of $H(X)$. There are some Banach spaces of functions which are not functional Banach spaces, but there are quite a few which are functional Banach spaces. We shall introduce them later in this section.

In case $H(X)$ is a functional Hilbert space, using the Riesz-representation theorem for every $x \in X$ we can find a unique $f_x \in H(X)$ such that

$$g(x) = \delta_x(g) = \langle g, f_x \rangle$$

for every $g \in H(X)$. The function f_x is called the kernel function of X induced by x. Let $K(X) = \{f_x \ : \ x \in X\}$. Then $K(X)$ is a subset of $H(X)$. The complex valued function K defined on $X \times X$ as

$$K(x,y) = \langle f_y, f_x \rangle$$

is called the reproducing kernel of $H(X)$. It is clear that

$$K(x,y) = \delta_x(f_y) = f_y(x)$$

and

$$\bar{K}(x,y) = \delta_y(f_x) = f_x(y)$$

for every x and y in X. If $\{e_j \ : \ j \in J\}$ is an orthonormal basis for $H(X)$, then the reproducing kernel K of $H(X)$ is given by

$$K(x,y) = \sum_{j\in J}{}' e_j(x)\overline{e_j(y)} \qquad \text{[See [137] for details].}$$

Examples

The following are some of the familiar examples of functional Banach spaces.

(1.3.1) ℓ^p- SPACES

Let X be any (countable) set and let m be the counting measure defined on the power set of X. Then $L^p(m)$, denoted as $\ell^p(X)$ is a functional Banach space for $1 \le p \le \infty$. The continuity of the evaluation functionals follows from the fact that

$$|\delta_x(f)| = |f(x)| \le \|f\|.$$

In particular the unitary space $\mathbb{C}^n$ and the classical sequence space ℓ^p are functional Banach spaces. In case $p = 2$, $\ell^p(X)$ is a functional Hilbert space and the reproducing kernel of $\ell^2(X)$ is given by

$$K(x,y) = \begin{cases} 1, & x = y \\ 0, & x \ne y. \end{cases}$$

Thus the reproducing kernel of ℓ^2 is the characteristic function of the diagonal of $\mathbb{N} \times \mathbb{N}$.

(1.3.2) H^p-SPACES

Let $X = D$, the open unit disk in the complex plane and let $1 \le p < \infty$. Let $H^p(D)$ denote the vector space of holomorphic functions f on D such that

$$\sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta \right\} < \infty.$$

Then it is well-known that $H^p(D)$ is a Banach space under the norm defined as

$$\|f\| = \left\{ \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta \right\}^{\frac{1}{p}}.$$

If $f \in H^p(D)$ and $z \in D$, then we know from [99] that

$$|\delta_z(f)| = |f(z)| \le \frac{2^{1/p}\|f\|}{(1-|z|)^{1/p}}.$$

Hence δ_z is continuous for every $z \in D$. Thus $H^p(D)$ is a functional Banach space. In case $p = 2$, $H^p(D)$ is a functional Hilbert space and the reproducing kernel of $H^2(D)$, known as Szego kernel, is given by

$$K(x, y) = 1/(1 - x\bar{y}),$$

where x and y vary in D [137].

Let $n \in \mathbb{N}$ and let $X = D^n$, the Cartesian product of n copies of the disk D. Let $H^p(D^n)$ denote the vector space of all those holomorphic functions f on D^n such that

$$\|f\|^p = \left\{\sup_{0<r<1} \int_{(\partial D)^n} |f(rw)|^p \, dm_n(w)\right\} < \infty ,$$

where m_n is the normalized Lebesgue measure on $(\partial D)^n$, ∂D denoting the boundary of D. Then $H^p(D^n)$ is a functional Banach space since

$$|f(z)| \le \|f\| \prod_{k=1}^{n} \left(1-|z_k|^2\right)^{-\frac{1}{p}}$$

for $z = (z_1, z_2, \ldots, z_n) \in D^n$. For details see Rudin [287], and Singh and Sharma [359]. $H^2(D^n)$ is a functional Hilbert space. Similarly, if $X = D_n$, the unit ball of $\mathbb{C}^n$, then space $H^p(D_n)$ of all those holomorphic functions f on D_n, for which

$$\|f\|^p = \sup_{0<r<1} \left\{\int_{\partial D_n} |f(rw)|^p \, d\sigma(w)\right\} < \infty ,$$

where σ is the rotation invariant Borel probability measure on the boundary ∂D_n of D_n, is a functional Banach space. We refer readers to Rudin [289] for further details about these spaces.

Let $X = P^+$, the upper-half plane and $H^p(P^+)$ denote the vector space of all those holomorphic functions f on P^+ for which

$$\| f \|^p = \sup_{y>0} \left\{ \int_{-\infty}^{\infty} | f(x+iy) |^p \, dx \right\} < \infty .$$

Then with this norm $H^p(P^+)$ is a functional Banach space since

$$|\delta_w(f)| = |f(w)| \le \frac{\|f\|}{(2\pi y)^{1/p}} ,$$

where $w = x + iy \in P^+$. For details see Duren [99] or Hoffman [146]. $H^2(P^+)$ is a functional Hilbert space and the reproducing kernel K is given by

$$K(w, u) = i / 2\pi(w - \bar{u}).$$

Note. All these spaces given in the set of examples are known as Hardy spaces and have very rich structures on them. More general Hardy spaces have been studied, but they are not presented here as we are not studying composition operators on them in this monograph.

(1.3.3) BERGMAN SPACES

Let $X = G$, a non-empty open connected subset of $\mathbb{C}$ and let m be the area Lebesgue measure on G. Let $A^p(G)$ be the linear space of all holomorphic functions f on G such that

$$\| f \|^p = \int_G | f |^p \, dm < \infty .$$

With the above norm $A^p(G)$ becomes a functional Banach space and for $p = 2$, $A^p(G)$ is a functional Hilbert space, where the inner product is given by

$$\langle f, g \rangle = \int_G f \bar{g} \, dm .$$

These spaces $A^p(G)$ for $1 \le p < \infty$ are known as Bergman spaces. In case $G = D$, the reproducing kernel of $A^2(D)$ is given by

$$K(x, y) = (1 / \pi) \left(1 / (1 - x\bar{y})^2 \right)$$

for $x, y \in D$. This kernel is known as the Bergman kernel.