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APPLIED AND INDUSTRIAL MATHEMATICS IN ITALY

Edited by
Mario Primicerio
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Vanda Valente

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Edited by

Mario Primicerio
Università di Firenze, Italy

Renato Spigler
Università di Roma 3, Italy

Vanda Valente
IAC-CNR, Roma, Italy

World Scientific

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PREFACE

In this volume we present 52 papers, which provide an overview of the research activity currently being pursued in Italy in the field of Applied and Industrial Mathematics, intended in the broadest sense. In fact a variety of areas is covered, as well as a large number of applications to engineering, finance, material science, environment, biology, etc. From the point of view of the mathematical content, the papers range from rigorous and formal analytical results to computational techniques, from modeling to engineering-oriented simulations.

All such contributions were solicited, refereed, and finally selected, to give an account of the state-of-the-art of the research work currently conducted in these fields in Italy. This has been done in the framework of the activities of the Italian Society for Applied and Industrial Mathematics (SIMAI), intended to promote and stimulate research in applied mathematics and its interaction with the area of industrial production.

These articles have been selected from a larger number of submitted papers by the Scientific Committee of SIMAI, consisting of Ubaldo Barberis, Enrico De Bernardis, Franco Brezzi, Giorgio Fotia, Luigia Puccio, and of the Editors of this book. We wish to thank all of them as well as the referees who have been involved in the selection procedure. Special thanks are also due to Paolo Rughetti who assisted us in preparing the lay-out of the volume.

The Editors
Roma, February 15th, 2005

CONTENTS

RESTRICTION MATRICES AND SYMMETRIC PANEL CLUSTERING METHOD FOR MULTI-DOMAIN SGBEM

A. AIMI AND M. DILIGENTI
Department of Mathematics, University of Parma
via M. D'Azeglio 85,
46100 Parma, Italy
E-mail: mauro.diligenti@unipr.it

A. SALVADORI
Department of Civil Engineering, University of Brescia
via Branze 38,
25123 Brescia, Italy

In the framework of the Symmetric Galerkin Boundary Element Method, in the past years different techniques have been proposed to reduce the computational cost of the Galerkin matrix evaluation. The dimension of the matrix can be significantly high in the case of problems defined on multi-domains. Here we couple the Panel Clustering Method with a set of Restriction Matrices developed to take computational advantage of possible symmetry properties of the integral problem.

1. SGBEM and Domain Decomposition Method

This section provides a very brief review of boundary integral equations for linear elliptic boundary value problems, their approximation via the symmetric-Galerkin method. The reader is asked to consult the cited references for further details.

Let $\Omega \subset \mathbb{R}^2$, be a bounded domain with a piecewise smooth boundary $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, where Γ_D and Γ_N are an open disjoint subset of Γ with $\Gamma_D \cap \Gamma_N = \emptyset$. We consider the mixed boundary value problem:

$$\begin{aligned} Lu(x) &= 0 && \text{for } x \in \Omega \\ u(x) &= \hat{u}(x) && \text{for } x \in \Gamma_D \quad \text{(Dirichlet condition)} \\ t(x) := (T_x u)(x) &= \hat{t}(x) && \text{for } x \in \Gamma_N \quad \text{(Neumann condition)} \end{aligned} \tag{1}$$

where $L(\cdot)$ is a linear elliptic partial differential operator of second order acting on u, describing the field equation inside the domain, $(T_x u)(x)$ the co-normal derivative of u for $x \in \Gamma$, $\hat{u}$ and $\hat{t}$ are given functions. Applications

of (1) are, e.g., boundary value problems in potential theory, in acoustic, elastostatics etc.. For problem (1) we may derive for the Cauchy data (u, t) a system of boundary integral equations (see [3]):

$$\frac{1}{2}\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} -K & V \\ D & K' \end{pmatrix}\begin{pmatrix} u \\ t \end{pmatrix}, \qquad \text{on } \Gamma, \tag{2}$$

using the single-layer potential

$$(Vt)(x) := \int_\Gamma U(x,y)t(y)\,ds_y\,, \qquad V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma);$$

the double-layer potential

$$(Ku)(x) := \int_\Gamma T_y U(x,y)u(y)\,ds_y\,, \qquad K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma);$$

the adjoint double-layer potential

$$(K't)(x) := \int_\Gamma T_x U(x,y)t(y)\,ds_y\,, \qquad K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma);$$

as well as the hyper-singular integral operator

$$(Du)(x) := -\int_\Gamma T_x T_y U(x,y)u(y)\,ds_y\,, \qquad D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma);$$

where $H^\alpha(\Gamma)$ denotes the Sobolev space with norm: $\|\cdot\|_{\alpha,\Gamma}$ and $\alpha \in \mathbb{R}$ (see [8]). The operators K and K' are defined by Cauchy singular integrals and D is defined by a hyper-singular finite part integral in the sense of Hadamard ([10]), due to the respective integral kernel singularity. The definition of all these boundary potentials is based on a fundamental solution $U(x,y)$ of the operator $L(\cdot)$. The mapping properties of all boundary integral operators defined above are well known ([3], [4]). Now, if we rewrite the first boundary integral equation in (2) for $x \in \Gamma_D$ and the second one for $x \in \Gamma_N$, we obtain a system of two BIEs of the first kind for the unknown Cauchy data u on Γ_N and t on Γ_D, of the form:

$$\begin{pmatrix} V_{DD} & -K_{ND} \\ K'_{DN} & D_{NN} \end{pmatrix}\begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} -V_{ND} & \frac{1}{2}I + K_{DD} \\ \frac{1}{2}I - K'_{NN} & -D_{DN} \end{pmatrix}\begin{pmatrix} \hat{t} \\ \hat{u} \end{pmatrix}, \tag{3}$$

where subscripts ab mean integration over Γ_a and evaluation over Γ_b, $a, b = D, N$. If we substitute in (3) $t = t^* + \bar{t}$, $u = u^* + \bar{u}$, where $\bar{t} \in H^{-1/2}(\Gamma)$ is an extension of the given Neumann data $\hat{t}$ with 0 on Γ_D and $\bar{u} \in H^{1/2}(\Gamma)$ is an extension of the given Dirichlet data $\hat{u}$ with 0 on Γ_N, then we obtain a system of boundary integral equations:

$$\begin{pmatrix} V_{DD} & -K_{ND} \\ K'_{DN} & D_{NN} \end{pmatrix}\begin{pmatrix} t^* \\ u^* \end{pmatrix} = \begin{pmatrix} -V_{ND} & \frac{1}{2}I + K_{DD} \\ \frac{1}{2}I - K'_{NN} & -D_{DN} \end{pmatrix}\begin{pmatrix} \hat{t} \\ \hat{u} \end{pmatrix} =: \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{4}$$

for the unknown functions $(u^*, t^*) \in \mathcal{S}(\Gamma) := \tilde{H}^{1/2}(\Gamma_N) \times \tilde{H}^{-1/2}(\Gamma_D)$. Here, the Sobolev spaces $\tilde{H}^{1/2}(\Gamma_N)$ and $\tilde{H}^{-1/2}(\Gamma_D)$ are defined in the usual way (see [8]) and the norm in $\mathcal{S}(\Gamma)$ is given by:

$$\|(u,t)\|^2_{\tilde{H}^{1/2}(\Gamma_N)\times\tilde{H}^{-1/2}(\Gamma_D)} := \|u\|^2_{H^{1/2}(\Gamma)} + \|t\|^2_{H^{-1/2}(\Gamma)}$$

The bilinear form:

$$\begin{aligned} b((u,t);(\phi,\psi)) := \; & (Vt,\psi)_{L^2(\Gamma_D)} - (Ku,\psi)_{L^2(\Gamma_D)} + \\ & (K't,\phi)_{L^2(\Gamma_N)} + (Du,\phi)_{L^2(\Gamma_N)}, \end{aligned} \tag{5}$$

is bounded and $\mathcal{S}(\Gamma)$-elliptic due to the mapping properties of the boundary integral operators (see [4]). Hence, the variational formulation:

$$b((u,t);(\phi,\psi)) = \mathcal{F}(\phi,\psi), \qquad \text{for all } (\phi,\psi) \in \mathcal{S}(\Gamma) \tag{6}$$

with the linear form: $\mathcal{F}(\phi,\psi) = (f_1,\psi)_{L^2(\Gamma_D)} + (f_2,\phi)_{L^2(\Gamma_N)}$, is uniquely solvable due to the Lax-Milgram theorem.
Let us introduce the finite-dimensional subspaces:

$$U_h = \operatorname{span}\{\varphi_k^{p_t}\}_{k=1}^{N_{\Gamma_D}} \subset \tilde{H}^{-1/2}(\Gamma_D), \quad W_h = \operatorname{span}\{\psi_k^{p_u}\}_{k=1}^{N_{\Gamma_N}} \subset \tilde{H}^{1/2}(\Gamma_N),$$

with piecewise trial functions of degree p_t and p_u, respectively. These spaces are related to a mesh $\mathcal{T} = \{e_1, \cdots, e_{N_h}\}$ defined on Γ such that $\Gamma = \cup_{i=1}^{N_h} \bar{e}_i$. For the following we have to assume approximation properties of the trial spaces U_h and W_h. From the mesh we require only, that the mesh ratio of two neighbouring elements is bounded by a constant. The Galerkin discretization of (6) is to find $(u_h, t_h) \in W_h \times U_h$ such that:

$$b((u_h,t_h),(\phi_h,\psi_h)) = \mathcal{F}(\phi_h,\psi_h) \quad \text{is satisfied for all} \quad (\phi_h,\psi_h) \in W_h \times U_h\,. \tag{7}$$

The finite-dimensional variational problem (7) is equivalent to a symmetric system of linear equations:

$$A\xi = b\,, \qquad \dim A = n = n(h, p_t, p_u) \tag{8}$$

Now, we will consider two sub-domains only, in order to simplify the formalism, occupying the domains Ω_1 and Ω_2, bounded by boundaries Γ^1, Γ^2 with the outward unit normal $\mathbf{n}^1$ and $\mathbf{n}^2$, respectively, and connected by an interface $\Gamma_c = \Gamma^1 \cap \Gamma^2$. The extension to more sub-domains, if not immediately obvious, is nevertheless seen to be relatively straightforward after a little thought. On the common boundary or interface Γ_c the matching conditions read:

$$u_c^1(x) = u_c^2(x) \equiv u_c(x), \qquad t_c^1(x) = -t_c^2(x) \equiv t_c(x) \tag{9}$$

The unknown functions are on Γ_N^r, Γ_D^r, and Γ_c: $\xi^r \equiv (t^r, u^r)$, $r = 1,2$, $\xi^c \equiv (t^c, u^c)$.

Let us write the system of boundary integral equations (4) to keep the interface Γ_c separate as a special part of the boundary of each sub-domain r and allowing for interface conditions. With the notation (9), these equations can be given in the following compact form:

$$\mathcal{L}_r^r \xi^r + \mathcal{L}_r^c \xi^c = \overline{f}^r, \qquad r = 1,2, \tag{10}$$

where we have set:

$$\mathcal{L}_r^r \equiv \begin{pmatrix} V_{DD}^r & -K_{ND}^r \\ K_{DN}^{\prime r} & D_{NN}^r \end{pmatrix}; \qquad \mathcal{L}_r^c \equiv \begin{pmatrix} (-1)^{r-1} V_{cD}^r & -K_{cD}^r \\ (-1)^r K_{cN}^{\prime r} & D_{cN}^r \end{pmatrix} \qquad r = 1,2. \tag{11}$$

The right-hand side $\overline{f}^r$ of (10), consists of the data functions f_1^r and f_2^r defined in (4), respectively, written for $r = 1,2$. Consider Γ_c as a part of the boundary of sub-domain 1 and 2 separately. Thus if we express both u^c and t^c along Γ_c and we make an appropriate combination of equations using (9), we obtain the integral equations for the common boundary Γ_c in the form:

$$\mathcal{L}_c^1 \xi^1 + \mathcal{L}_c^2 \xi^2 + \mathcal{L}_c^c \xi^c = \overline{f}^c, \tag{12}$$

where:

$$\mathcal{L}_c^r \equiv \begin{pmatrix} (-1)^{r-1} V_{Dc}^r & (-1)^r K_{Nc}^r \\ -K_{Dc}^{\prime r} & D_{Nc}^r \end{pmatrix}, \quad r = 1,2; \tag{13}$$

$$\mathcal{L}_c^c \equiv \begin{pmatrix} V_{cc}^1 + V_{cc}^2 & -(K_{cc}^1 - K_{cc}^2) \\ K_{cc}^{\prime 1} - K_{cc}^{\prime 2} & D_{cc}^1 + D_{cc}^2 \end{pmatrix}. \tag{14}$$

Consider the set of (10) and (12) which, together, govern the solution u on Γ_N^1, Γ_N^2, Γ_c and t on Γ_D^1, Γ_D^2 and Γ_c. The global system, taking into account (10)-(14), reads:

$$\begin{pmatrix} \mathcal{L}_1^1 & O & \mathcal{L}_1^c \\ O & \mathcal{L}_2^2 & \mathcal{L}_2^c \\ \mathcal{L}_c^1 & \mathcal{L}_c^2 & \mathcal{L}_c^c \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^c \end{pmatrix} = \begin{pmatrix} \overline{f}^1 \\ \overline{f}^2 \\ \overline{f}^c \end{pmatrix}. \tag{15}$$

The Galerkin discretization of the symmetric formulation (15) clearly generates a block linear system similar to system (8).

When the domain Ω and therefore its boundary have some geometrical symmetries the computational cost of the SGBEM matrix generation and the linear system solution can be significantly lowered. This can be done by using suitable restriction matrices.

2. Restriction Matrices and Symmetric PCM

Let $\mathcal{G}$ be a finite group of t congruences having at least two distinct elements of the Euclidean space $\mathbb{R}^m$, $m = 2, 3$. The group $\mathcal{G}$ can be described by orthogonal matrices $\gamma_1, \gamma_2, \cdots, \gamma_t$ of order m; let γ_1 be the identity matrix. From the theory of group representations it follows that any finite group $\mathcal{G}$ admits a finite number q of unitary irreducible, pairwise not-equivalent matrix representations:

$$\{\omega^{(1)}(\gamma_i)\},\ \{\omega^{(2)}(\gamma_i)\},\ \cdots,\ \{\omega^{(q)}(\gamma_i)\}, \qquad i = 1, \cdots, t\,. \tag{16}$$

Let d_ℓ be the order of the representation $\{\omega^{(\ell)}(\gamma_i)\}$, i.e. the order of the matrices $\omega^{(\ell)}(\gamma_i)$. The number q of the representations (16) and the orders d_ℓ, $\ell = 1, \cdots, q$, only depend on $\mathcal{G}$. Having set:

$$M = d_1 + d_2 + \cdots + d_q\,, \tag{17}$$

from the theory of group representation it follows that $q \leq M \leq t$ and $q = M = t$ if and only if $\mathcal{G}$ is an Abelian group. Matrices depending only on group $\mathcal{G}$ and on the system of representations (16) called *elementary restriction matrices*, were introduced in [2]. For vector problems, these matrices have the following simple definition:

$$\mathcal{E}_{\ell k} = \sqrt{\frac{d_\ell}{t}} \begin{pmatrix} \omega^{(\ell)}_{k1}(\gamma_1)\gamma_1^* & \omega^{(\ell)}_{k1}(\gamma_2)\gamma_2^* & \cdots & \omega^{(\ell)}_{k1}(\gamma_t)\gamma_t^* \\ \omega^{(\ell)}_{k2}(\gamma_1)\gamma_1^* & \omega^{(\ell)}_{k2}(\gamma_2)\gamma_2^* & \cdots & \omega^{(\ell)}_{k2}(\gamma_t)\gamma_t^* \\ \cdots & \cdots & \cdots & \cdots \\ \omega^{(\ell)}_{kd_\ell}(\gamma_1)\gamma_1^* & \omega^{(\ell)}_{kd_\ell}(\gamma_2)\gamma_2^* & \cdots & \omega^{(\ell)}_{kd_\ell}(\gamma_t)\gamma_t^* \end{pmatrix}, \quad \begin{matrix} \ell = 1, \cdots, q \\ k = 1, \cdots, d_\ell \end{matrix} \tag{18}$$

where γ_i^* denotes the transpose of γ_i. The total number of these matrices is M, with M given by (17). Due to the orthogonality properties of the representations, matrix $\mathcal{E}_{\ell k}$ has pairwise orthogonal rows. Therefore the rank of matrix $\mathcal{E}_{\ell k}$ is md_ℓ.

Let $\mathcal{R}_s$, $s = 1, \cdots, p$, be full rank real matrices with, respectively n_s rows and n columns such that $n = \sum_{s=1}^{p} n_s$ and:

$$\mathcal{R}_s\mathcal{R}_s^* = I_s, \qquad \mathcal{R}_s\mathcal{R}_{s'}^* = O,\ s \neq s', \qquad \sum_{s=1}^{p} \mathcal{R}_s^*\mathcal{R}_s = I \tag{19}$$

where I_s, I are identity matrices of order n_s and n, respectively. In literature, these matrices are called restriction matrices and have been widely used in the context of parallel multi-grid algorithms, and domain decomposition reduction methods for a numerical solution of elliptic boundary value problems ([5]). The following result holds ([2]).

Theorem 2.1. *Let $A\xi = b$ be a linear system, where A is a non-singular matrix of order n and $\mathcal{R}_1, \cdots, \mathcal{R}_p$, p restriction matrices verifying condition* (19). *If matrix A verifies conditions:*

$$\mathcal{R}_k A \mathcal{R}_j^* = 0, \qquad k \neq j, \qquad k, j = 1, \cdots, p \tag{20}$$

then the system $A\xi = b$ can be decomposed in p independent subsystems

$$A_\ell \xi_\ell = \mathcal{R}_\ell b, \tag{21}$$

with $A_\ell = \mathcal{R}_\ell A \mathcal{R}_\ell^*$ a non-singular matrix of dimension $n_\ell \times n_\ell$. The solution ξ is obtained by $\xi = \sum_{\ell=1}^{p} \mathcal{R}_\ell^* \xi_\ell$, where ξ_ℓ is the solution of (21).

The problem of decomposition of the system (8) is then reduced to the construction of a suitable system of restriction matrices verifying (20). Starting from *elementary restriction matrices* defined in (18), it is possible to construct M restriction matrices $\mathcal{R}_{\ell k}$, $\ell = 1, \cdots, q$; $k = 1, \cdots, d_\ell$ related to the space $U_h \times W_h$ for the splitting of the SGBEM linear system when the problem is *$\mathcal{G}$-invariant* (see [2]). We observe that the linear subsystem generation does not imply the construction of the whole BEM matrix A and has a cost in terms of *flops* of order n^2. The reduction factor in solving all linear subsystems instead of $A\xi = b$ with a same direct method is $O(1/t^3 \sum_{\ell=1}^{q} d_\ell^3)$. Decomposition algorithms and computational details can be found in [2].

The matrix elements of the dense linear symmetric system (8) are double integrals with weakly singular, singular or hyper-singular kernels. The singularity of integral operators arise in the evaluation only of a number of elements of order n along the diagonal band of A. Their generation can be made with suitable quadrature rules ([1]); the evaluation of all the remaining matrix elements does not present any difficulty, but can have a heavy computational cost when n is high. In this context, many different techniques have been proposed during these last years to reduce the computational cost of the Galerkin matrix evaluation and the linear system solution. Among them, we recall the Panel Clustering Method (PCM), that was introduced firstly by Hackbush and Nowak ([7]) and further developed by various authors ([6], [9]). Here we show how this technique can be used in connection with restriction matrices under hypothesis of problem invariance. The PCM allows a fast evaluation of the most off-diagonal elements of the boundary element matrix and can be basically summarized in the following three main steps:

(i) a hierarchical structuring of the boundary mesh $\mathcal{T}$, called *cluster tree*;
(ii) a visiting of the cluster tree with a suitable *η-admissibility* condition;

(iii) an approximation of the most off-diagonal elements of the Galerkin matrix, related to **(ii)** and based on a convenient truncated expansion of the kernel functions.

For the first step **(i)** we recall that: a cluster tree $\mathcal{C}$ is a hierarchy of clusters, starting from the *root* $\mathcal{T} = \tau_R$, ordered with respect to the relation of the inclusion, where a cluster τ is a non empty union of elements $e_i \in \mathcal{T}$, equipped with a center x_τ and a radius ρ_τ such that the ball $B_{\rho_\tau}(x_\tau) = \{x \in \mathbb{R}^m : \| x - x_\tau \|_2 \leq \rho_\tau\}$ is the smallest ball containing τ. Note that at each level of the hierarchy the cluster union coincides with $\mathcal{T}$. Different algorithms for the construction of cluster trees have been proposed (leading e.g. to a balanced binary tree ([6], [7], [9]). Unfortunately these cluster trees do not share the symmetry properties of boundary Γ and mesh $\mathcal{T}$. If $\mathcal{T}$ is a $\mathcal{G}$-invariant mesh, this property can be exploited in order both to reduce computer time in generating cluster trees and to lower memory requirement for data storage of the tree structure. Then, let $\mathcal{T}$ be a $\mathcal{G}$-invariant mesh: this means that there is a subset of mesh elements, in the sequel indicated with $\tilde{\mathcal{T}}$, which, through the action of the congruences of $\mathcal{G}$, generates the whole $\mathcal{T}$, i.e. $\mathcal{T} = \bigcup_{i=1}^{t} \gamma_i(\tilde{\mathcal{T}})$, interpreting the notation $\gamma_i(\tilde{\mathcal{T}})$ as the application of the i-th congruence of group $\mathcal{G}$ to every mesh element belonging to $\tilde{\mathcal{T}}$. Further, let $\tilde{\mathcal{C}}$ be a cluster tree related to $\tilde{\mathcal{T}}$: a cluster tree for the entire mesh $\mathcal{T}$ can be recovered as $\mathcal{C} = \bigcup_{i=1}^{t} \gamma_i(\tilde{\mathcal{C}})$. If τ is a cluster of $\mathcal{C}$, we will find in $\mathcal{C}$ all the images $\gamma_i(\tau)$, $i = 2, \cdots, t$; these clusters satisfy the properties: $x_{\gamma_i(\tau)} = \gamma_i x_\tau$, $\rho_{\gamma_i(\tau)} = \rho_\tau$.

For the second step **(ii)** of PCM we introduce the following η-admissibility condition: given $0 < \eta_0 < 1$ and fixed $0 < \eta < \eta_0$, a pair of clusters $(\tau, \tau') \in \mathcal{C} \times \mathcal{C}$, with radius ρ_τ, $\rho_{\tau'}$ respectively, is said η-admissible if:

$$\max\{\rho_\tau, \rho_{\tau'}\} \leq \eta \, \mathrm{dist}(\tau, \tau') \tag{22}$$

where $\mathrm{dist}(\tau, \tau')$ denotes the distance of the ball B_{ρ_τ} from the ball $B_{\rho_{\tau'}}$. The visiting of $\mathcal{C} \times \mathcal{C}$ with the above η-admissibility condition produces pairs of η-admissible clusters, hence pairs of sets of mesh element indices that generate a subdivision of the off-diagonal band of the Galerkin matrix A into η-admissible blocks (a sort of grid which covers A), since the matrix element indices are obviously related to mesh element indices.
The choice of the η_0 parameter is related to the kernel function to be approximated, in order to ensure the convergence of PCM.

In the third step **(iii)** of PCM, the elements of the η-admissible blocks of matrix A related to the η-admissible pairs $(\tau, \tau') \in \mathcal{C} \times \mathcal{C}$ with centers x_τ