Graduate Texts in Mathematics

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continued after Index

John B. Conway

Functions of One Complex Variable

Second Edition

With 30 Illustrations

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To
Ann

PREFACE

This book is intended as a textbook for a first course in the theory of functions of one complex variable for students who are mathematically mature enough to understand and execute $\epsilon - \delta$ arguments. The actual prerequisites for reading this book are quite minimal; not much more than a stiff course in basic calculus and a few facts about partial derivatives. The topics from advanced calculus that are used (e.g., Leibniz's rule for differentiating under the integral sign) are proved in detail.

Complex Variables is a subject which has something for all mathematicians. In addition to having applications to other parts of analysis, it can rightly claim to be an ancestor of many areas of mathematics (e.g., homotopy theory, manifolds). This view of Complex Analysis as "An Introduction to Mathematics" has influenced the writing and selection of subject matter for this book. The other guiding principle followed is that all definitions, theorems, etc. should be clearly and precisely stated. Proofs are given with the student in mind. Most are presented in detail and when this is not the case the reader is told precisely what is missing and asked to fill in the gap as an exercise. The exercises are varied in their degree of difficulty. Some are meant to fix the ideas of the section in the reader's mind and some extend the theory or give applications to other parts of mathematics. (Occasionally, terminology is used in an exercise which is not defined—e.g., group, integral domain.)

Chapters I through V and Sections VI.1 and VI.2 are basic. It is possible to cover this material in a single semester only if a number of proofs are omitted. Except for the material at the beginning of Section VI.3 on convex functions, the rest of the book is independent of VI.3 and VI.4.

Chapter VII initiates the student in the consideration of functions as points in a metric space. The results of the first three sections of this chapter are used repeatedly in the remainder of the book. Sections four and five need no defense; moreover, the Weierstrass Factorization Theorem is necessary for Chapter XI. Section six is an application of the factorization theorem. The last two sections of Chapter VII are not needed in the rest of the book although they are a part of classical mathematics which no one should completely disregard.

The remaining chapters are independent topics and may be covered in any order desired.

Runge's Theorem is the inspiration for much of the theory of Function Algebras. The proof presented in section VIII.1 is, however, the classical one involving "pole pushing". Section two applies Runge's Theorem to obtain a more general form of Cauchy's Theorem. The main results of sections three and four should be read by everyone, even if the proofs are not.

Chapter IX studies analytic continuation and introduces the reader to analytic manifolds and covering spaces. Sections one through three can be considered as a unit and will give the reader a knowledge of analytic continuation without necessitating his going through all of Chapter IX.

Chapter X studies harmonic functions including a solution of the Dirichlet Problem and the introduction of Green's Function. If this can be called applied mathematics it is part of applied mathematics that everyone should know.

Although they are independent, the last two chapters could have been combined into one entitled "Entire Functions". However, it is felt that Hadamard's Factorization Theorem and the Great Theorem of Picard are sufficiently different that each merits its own chapter. Also, neither result depends upon the other.

With regard to Picard's Theorem it should be mentioned that another proof is available. The proof presented here uses only elementary arguments while the proof found in most other books uses the modular function.

There are other topics that could have been covered. Some consideration was given to including chapters on some or all of the following: conformal mapping, functions on the disk, elliptic functions, applications of Hilbert space methods to complex functions. But the line had to be drawn somewhere and these topics were the victims. For those readers who would like to explore this material or to further investigate the topics covered in this book, the bibliography contains a number of appropriate entries.

Most of the notation used is standard. The word "iff" is used in place of the phrase "if and only if", and the symbol ■ is used to indicate the end of a proof. When a function (other than a path) is being discussed, Latin letters are used for the domain and Greek letters are used for the range.

This book evolved from classes taught at Indiana University. I would like to thank the Department of Mathematics for making its resources available to me during its preparation. I would especially like to thank the students in my classes; it was actually their reaction to my course in Complex Variables that made me decide to take the plunge and write a book. Particular thanks should go to Marsha Meredith for pointing out several mistakes in an early draft, to Stephen Berman for gathering the material for several exercises on algebra, and to Larry Curnutt for assisting me with the final corrections of the manuscript. I must also thank Ceil Sheehan for typing the final draft of the manuscript under unusual circumstances.

Finally, I must thank my wife to whom this book is dedicated. Her encouragement was the most valuable assistance I received.

John B. Conway

PREFACE FOR THE SECOND EDITION

I have been very pleased with the success of my book. When it was apparent that the second printing was nearly sold out, Springer-Verlag asked me to prepare a list of corrections for a third printing. When I mentioned that I had some ideas for more substantial revisions, they reacted with characteristic enthusiasm.

There are four major differences between the present edition and its predecessor. First, John Dixon's treatment of Cauchy's Theorem has been included. This has the advantage of providing a quick proof of the theorem in its full generality. Nevertheless, I have a strong attachment to the homotopic version that appeared in the first edition and have proved this form of Cauchy's Theorem as it was done there. This version is very geometric and quite easy to apply. Moreover, the notion of homotopy is needed for the later treatment of the monodromy theorem; hence, inclusion of this version yields benefits far in excess of the time needed to discuss it.

Second, the proof of Runge's Theorem is new. The present proof is due to Sandy Grabiner and does not use "pole pushing". In a sense the "pole pushing" is buried in the concept of uniform approximation and some ideas from Banach algebras. Nevertheless, it should be emphasized that the proof is entirely elementary in that it relies only on the material presented in this text.

Next, an Appendix B has been added. This appendix contains some bibliographical material and a guide for further reading.

Finally, several additional exercises have been added.

There are also minor changes that have been made. Several colleagues in the mathematical community have helped me greatly by providing constructive criticism and pointing out typographical errors. I wish to thank publicly Earl Berkson, Louis Brickman, James Deddens, Gerard Keough, G. K. Kristiansen, Andrew Lenard, John Mairhuber, Donald C. Meyers, Jeffrey Nunemacher, Robert Olin, Donald Perlis, John Plaster, Hans Sagan, Glenn Schober, David Stegenga, Richard Varga, James P. Williams, and Max Zorn.

Finally, I wish to thank the staff at Springer-Verlag New York not only for their treatment of my book, but also for the publication of so many fine books on mathematics. In the present time of shrinking graduate enrollments and the consequent reluctance of so many publishers to print advanced texts and monographs, Springer-Verlag is making a contribution to our discipline by increasing its efforts to disseminate the recent developments in mathematics.

John B. Conway

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Chapter I

The Complex Number System

§1. *The real numbers*

We denote the set of all real numbers by $\mathbb{R}$. It is assumed that each reader is acquainted with the real number system and all its properties. In particular we assume a knowledge of the ordering of $\mathbb{R}$, the definitions and properties of the supremum and infimum (sup and inf), and the completeness of $\mathbb{R}$ (every set in $\mathbb{R}$ which is bounded above has a supremum). It is also assumed that every reader is familiar with sequential convergence in $\mathbb{R}$ and with infinite series. Finally, no one should undertake a study of Complex Variables unless he has a thorough grounding in functions of one real variable. Although it has been traditional to study functions of several real variables before studying analytic function theory, this is not an essential prerequisite for this book. There will not be any occasion when the deep results of this area are needed.

§2. *The field of complex numbers*

We define $\mathbb{C}$, the complex numbers, to be the set of all ordered pairs (a, b) where a and b are real numbers and where addition and multiplication are defined by:

$$(a, b)+(c, d) = (a+c, b+d)$$

$$(a, b)\,(c, d) = (ac-bd, bc+ad)$$

It is easily checked that with these definitions $\mathbb{C}$ satisfies all the axioms for a field. That is, $\mathbb{C}$ satisfies the associative, commutative and distributive laws for addition and multiplication; $(0,0)$ and $(1,0)$ are identities for addition and multiplication respectively, and there are additive and multiplicative inverses for each nonzero element in $\mathbb{C}$.

We will write a for the complex number $(a, 0)$. This mapping $a \to (a, 0)$ defines a field isomorphism of $\mathbb{R}$ into $\mathbb{C}$ so we may consider $\mathbb{R}$ as a subset of $\mathbb{C}$. If we put $i = (0, 1)$ then $(a, b) = a+bi$. From this point on we abandon the ordered pair notation for complex numbers.

Note that $i^2 = -1$, so that the equation $z^2+1 = 0$ has a root in $\mathbb{C}$. In fact, for each z in $\mathbb{C}$, $z^2+1 = (z+i)\,(z-i)$. More generally, if z and w are complex numbers we obtain

$$z^2+w^2 = (z+iw)\,(z-iw)$$

By letting z and w be real numbers a and b we can obtain (with both a and $b \neq 0$)

$$\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\left(\frac{b}{a^2+b^2}\right)$$

so that we have a formula for the reciprocal of a complex number.

When we write $z = a+ib$ $(a, b \in \mathbb{R})$ we call a and b the *real* and *imaginary parts* of z and denote this by $a = \operatorname{Re} z$, $b = \operatorname{Im} z$.

We conclude this section by introducing two operations on $\mathbb{C}$ which are not field operations. If $z = x+iy(x, y \in \mathbb{R})$ then we define $|z| = (x^2+y^2)^{\frac{1}{2}}$ to be the *absolute value* of z and $\bar{z} = x-iy$ is the *conjugate* of z. Note that

2.1 $$|z|^2 = z\bar{z}$$

In particular, if $z \neq 0$ then

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

The following are basic properties of absolute values and conjugates whose verifications are left to the reader.

2.2 $$\operatorname{Re} z = \tfrac{1}{2}(z+\bar{z}) \quad \text{and} \quad \operatorname{Im} z = \frac{1}{2i}(z-\bar{z}).$$

2.3 $$(\overline{z+w}) = \bar{z}+\bar{w} \quad \text{and} \quad \overline{zw} = \bar{z}\bar{w}.$$

2.4 $$|zw| = |z|\,|w|.$$

2.5 $$|z/w| = |z|/|w|.$$

2.6 $$|\bar{z}| = |z|.$$

The reader should try to avoid expanding z and w into their real and imaginary parts when he tries to prove these last three. Rather, use (2.1), (2.2), and (2.3).

Exercises

1. Find the real and imaginary parts of each of the following:

$$\frac{1}{z};\ \frac{z-a}{z+a}\,(a \in \mathbb{R});\ z^3;\ \frac{3+5i}{7i+1};\ \left(\frac{-1+i\sqrt{3}}{2}\right)^3;$$

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^6;\ i^n;\ \left(\frac{1+i}{\sqrt{2}}\right)^n \quad \text{for} \quad 2 \leq n \leq 8.$$

2. Find the absolute value and conjugate of each of the following:

$$-2+i;\ -3;\ (2+i)(4+3i);\ \frac{3-i}{\sqrt{2}+3i};\ \frac{i}{i+3};$$

$$(1+i)^6;\ i^{17}.$$

3. Show that z is a real number if and only if $z = \bar{z}$.
4. If z and w are complex numbers, prove the following equations:

$$|z+w|^2 = |z|^2 + 2\text{Re}\, z\bar{w} + |w|^2.$$

$$|z-w|^2 = |z|^2 - 2\text{Re}\, z\bar{w} + |w|^2.$$

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

5. Use induction to prove that for $z = z_1 + \ldots + z_n$; $w = w_1 w_2 \ldots w_n$: $|w| = |w_1| \ldots |w_n|$; $\bar{z} = \bar{z}_1 + \ldots + \bar{z}_n$; $\bar{w} = \bar{w}_1 \ldots \bar{w}_n$.
6. Let $R(z)$ be a rational function of z. Show that $\overline{R(z)} = R(\bar{z})$ if all the coefficients in $R(z)$ are real.

§3. *The complex plane*

From the definition of complex numbers it is clear that each z in $\mathbb{C}$ can be identified with the unique point $(\text{Re}\, z, \text{Im}\, z)$ in the plane $\mathbb{R}^2$. The addition of complex numbers is exactly the addition law of the vector space $\mathbb{R}^2$. If z and w are in $\mathbb{C}$ then draw the straight lines from z and w to 0 $(=(0, 0))$. These form two sides of a parallelogram with 0, z and w as three vertices. The fourth vertex turns out to be $z+w$.

Note also that $|z-w|$ is exactly the distance between z and w. With this in mind the last equation of Exercise 4 in the preceding section states the *parallelogram law*: The sum of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of its diagonals.

A fundamental property of a distance function is that it satisfies the triangle inequality (see the next chapter). In this case this inequality becomes

$$|z_1 - z_2| \leq |z_1 - z_3| + |z_3 - z_2|$$

for complex numbers z_1, z_2, z_3. By using $z_1 - z_2 = (z_1 - z_3) + (z_3 - z_2)$, it is easy to see that we need only show

3.1 $$|z+w| \leq |z| + |w| \quad (z, w \in \mathbb{C}).$$

To show this first observe that for any z in $\mathbb{C}$,

3.2 $$-|z| \leq \text{Re}\, z \leq |z|$$
$$-|z| \leq \text{Im}\, z \leq |z|$$

Hence, $\text{Re}\,(z\bar{w}) \leq |z\bar{w}| = |z|\,|w|$. Thus,

$$\begin{aligned} |z+w|^2 &= |z|^2 + 2\text{Re}\,(z\bar{w}) + |w|^2 \\ &\leq |z|^2 + 2|z|\,|w| + |w|^2 \\ &= (|z| + |w|)^2, \end{aligned}$$

from which (3.1) follows. (This is called the *triangle inequality* because, if we represent z and w in the plane, (3.1) says that the length of one side of the triangle $[0, z, z+w]$ is less than the sum of the lengths of the other two sides. Or, the shortest distance between two points is a straight line.) On encountering an inequality one should always ask for necessary and sufficient conditions that equality obtains. From looking at a triangle and considering the geometrical significance of (3.1) we are led to consider the condition $z = tw$ for some $t \in \mathbb{R}$, $t \geq 0$. (or $w = tz$ if $w = 0$). It is clear that equality will occur when the two points are colinear with the origin. In fact, if we look at the proof of (3.1) we see that a necessary and sufficient condition for $|z+w| = |z|+|w|$ is that $|z\bar{w}| = \text{Re}\,(z\bar{w})$. Equivalently, this is $z\bar{w} \geq 0$ (i.e., $z\bar{w}$ is a real number and is non negative). Multiplying this by w/w we get $|w|^2(z/w) \geq 0$ if $w \neq 0$. If

$$t = z/w = \left(\frac{1}{|w|^2}\right) |w|^2(z/w)$$

then t ≥ 0 and $z = tw$.

By induction we also get

3.3 $$|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2|+\ldots+|z_n|$$

Also useful is the inequality

3.4 $$\big||z|-|w|\big| \leq |z-w|$$

Now that we have given a geometric interpretation of the absolute value let us see what taking a complex conjugate does to a point in the plane. This is also easy; in fact, $\bar{z}$ is the point obtained by reflecting z across the x-axis (i.e., the real axis).

Exercises

1. Prove (3.4) and give necessary and sufficient conditions for equality.
2. Show that equality occurs in (3.3) if and only if $z_k/z_l \geq 0$ for any integers k and l, $1 \leq k, l \leq n$, for which $z_l \neq 0$.
3. Let $a \in \mathbb{R}$ and $c > 0$ be fixed. Describe the set of points z satisfying

$$|z-a|-|z+a| = 2c$$

for every possible choice of a and c. Now let a be any complex number and, using a rotation of the plane, describe the locus of points satisfying the above equation.

§4. *Polar representation and roots of complex numbers*

Consider the point $z = x+iy$ in the complex plane $\mathbb{C}$. This point has polar coordinates (r, θ): $x = r\cos\theta$, $y = r\sin\theta$. Clearly $r = |z|$ and θ is the angle between the positive real axis and the line segment from 0 to z. Notice that θ plus any multiple of 2π can be substituted for θ in the above equations. The angle θ is called the *argument of* z and is denoted by $\theta = \arg z$. Because of the ambiguity of θ, "arg" is not a function. We introduce the notation

4.1 $$\text{cis}\,\theta = \cos\theta + i\sin\theta.$$

Let $z_1 = r_1 \operatorname{cis} \theta_1$, $z_2 = r_2 \operatorname{cis} \theta_2$. Then $z_1 z_2 = r_1 r_2 \operatorname{cis} \theta_1 \operatorname{cis} \theta_2 = r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1)]$. By the formulas for the sine and cosine of the sum of two angles we get

4.2 $$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

Alternately, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$. (What function of a real variable takes products into sums?) By induction we get for $z_k = r_k \operatorname{cis} \theta_k$, $1 \le k \le n$.

4.3 $$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n \operatorname{cis}(\theta_1 + \ldots + \theta_n)$$

In particular,

4.4 $$z^n = r^n \operatorname{cis}(n\theta),$$

for every integer $n \ge 0$. Moreover if $z \ne 0$, $z \cdot [r^{-1} \operatorname{cis}(-\theta)] = 1$; so that (4.4) also holds for all integers n, positive, negative, and zero, if $z \ne 0$. As a special case of (4.4) we get *de Moivre's formula*:

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

We are now in a position to consider the following problem: For a given complex number $a \ne 0$ and an integer $n \ge 2$, can you find a number z satisfying $z^n = a$? How many such z can you find? In light of (4.4) the solution is easy. Let $a = |a| \operatorname{cis} \alpha$; by (4.4), $z = |a|^{1/n} \operatorname{cis}(\alpha/n)$ fills the bill. However this is not the only solution because $z' = |a|^{1/n} \operatorname{cis} \frac{1}{n}(\alpha + 2\pi)$ also satisfies $(z')^n = a$. In fact each of the numbers

4.5 $$|a|^{1/n} \operatorname{cis} \frac{1}{n}(\alpha + 2\pi k),\ 0 \le k \le n-1,$$

in an nth root of a. By means of (4.4) we arrive at the following: for each non zero number a in $\mathbb{C}$ there are *n distinct* nth roots of a; they are given by formula (4.5).

Example

Calculate the nth roots of unity. Since $1 = \operatorname{cis} 0$, (4.5) gives these roots as

$$1, \operatorname{cis} \frac{2\pi}{n}, \operatorname{cis} \frac{4\pi}{n}, \ldots, \operatorname{cis} \frac{2\pi}{n}(n-1).$$

In particular, the cube roots of unity are

$$1, \frac{1}{\sqrt{2}}(-1 + i\sqrt{3}), \frac{1}{\sqrt{2}}(1 - i\sqrt{3}).$$

Exercises

1. Find the sixth roots of unity.