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CALCULUS

VOLUME 1

One-Variable Calculus, with an
Introduction to Linear Algebra

SECOND EDITION

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To
Jane and Stephen

PREFACE

Excerpts from the Preface to the First Edition

There seems to be no general agreement as to what should constitute a first course in calculus and analytic geometry. Some people insist that the only way to really understand calculus is to start off with a thorough treatment of the real-number system and develop the subject step by step in a logical and rigorous fashion. Others argue that calculus is primarily a tool for engineers and physicists; they believe the course should stress applications of the calculus by appeal to intuition and by extensive drill on problems which develop manipulative skills. There is much that is sound in both these points of view. Calculus is a deductive science and a branch of pure mathematics. At the same time, it is very important to remember that calculus has strong roots in physical problems and that it derives much of its power and beauty from the variety of its applications. It is possible to combine a strong theoretical development with sound training in technique; this book represents an attempt to strike a sensible balance between the two. While treating the calculus as a deductive science, the book does not neglect applications to physical problems. Proofs of all the important theorems are presented as an essential part of the growth of mathematical ideas; the proofs are often preceded by a geometric or intuitive discussion to give the student some insight into why they take a particular form. Although these intuitive discussions will satisfy readers who are not interested in detailed proofs, the complete proofs are also included for those who prefer a more rigorous presentation.

The approach in this book has been suggested by the historical and philosophical development of calculus and analytic geometry. For example, integration is treated before differentiation. Although to some this may seem unusual, it is historically correct and pedagogically sound. Moreover, it is the best way to make meaningful the true connection between the integral and the derivative.

The concept of the integral is defined first for step functions. Since the integral of a step function is merely a finite sum, integration theory in this case is extremely simple. As the student learns the properties of the integral for step functions, he gains experience in the use of the summation notation and at the same time becomes familiar with the notation for integrals. This sets the stage so that the transition from step functions to more general functions seems easy and natural.

Preface to the Second Edition

The second **edition** differs from the first in **many** respects. Linear algebra has been incorporated, the mean-value theorems and routine applications of **calculus** are introduced at an earlier stage, and **many** new and easier **exercises** have been added. A glance at the table of contents reveals that the book has been divided into smaller **chapters**, **each** centering on an important concept. Several sections have been rewritten and reorganized to **provide** better motivation and to improve the flow of ideas.

As in the first **edition**, a historical introduction **precedes** each important new concept, tracing its development from an early intuitive physical notion to its **precise** mathematical formulation. The student is told something of the struggles of the past and of the triumphs of the men who contributed most to the subject. Thus the student becomes an active participant in the evolution of ideas rather than a passive observer of results.

The second **edition**, like the first, is divided into two volumes. The first two thirds of Volume I deals with the **calculus** of functions of **one** variable, including **infinite** series and an introduction to differential equations. The last third of Volume I introduces linear algebra with applications to geometry and analysis. **Much** of this material **leans** heavily on the **calculus** for examples that illustrate the general theory. It **provides** a natural blending of algebra and analysis and helps pave the way for the transition from **one**-variable **calculus** to multivariable calculus, discussed in Volume II. Further development of linear algebra **will occur** as needed in the second **edition** of Volume II.

Once **again** I acknowledge with pleasure my debt to Professors H. F. Bohnenblust, A. Erdélyi, F. B. Fuller, K. Hoffman, G. Springer, and H. S. Zuckerman. Their influence on the first **edition** **continued** into the second. In preparing the second **edition**, I received additional help from Professor **Basil** Gordon, who suggested **many** improvements. Thanks are also due George Springer and William P. Ziemer, who read the final draft. The staff of the Blaisdell Publishing Company has, as always, been helpful; I appreciate their **sympathetic** **consideration** of my wishes concerning format and typography.

Finally, it gives me **special** pleasure to express my gratitude to my wife for the **many** ways she has contributed **during** the preparation of both editions. In grateful acknowledgment I happily dedicate this book to her.

T. M. A.

Pasadena, California
September 16, 1966

CONTENTS

1. INTRODUCTION

Part 1. Historical Introduction

11.1	The two basic concepts of calculus	1
1 1.2	Historical background	2
1 1.3	'The method of exhaustion for the area of a parabolic segment	3
*I 1.4	Exercises	8
1 1.5	A critical analysis of Archimedes' method	8
1 1.6	The approach to calculus to be used in this book	10

Part 2. Some Basic Concepts of the Theory of Sets

12.1	Introduction to set theory	11
1 2.2	Notations for designating sets	12
12.3	Subsets	12
1 2.4	Unions, intersections, complements	13
1 2.5	Exercises	15

Part 3. A Set of Axioms for the Real-Number System

13.1	Introduction	17
1 3.2	The field axioms	17
*I 3.3	Exercises	19
1 3.4	The order axioms	19
*I 3.5	Exercises	21
1 3.6	Integers and rational numbers	21

1 3.7	Geometric interpretation of real numbers as points on a line	22
1 3.8	Upper bound of a set, maximum element, least upper bound (supremum)	23
1 3.9	The least-Upper-bound axiom (completeness axiom)	25
1 3.10	The Archimedean property of the real-number system	25
1 3.11	Fundamental properties of the supremum and infimum	26
*1 3.12	Exercises	28
*1 3.13	Existence of square roots of nonnegative real numbers	29
*1 3.14	Roots of higher order. Rational powers	30
*1 3.15	Representation of real numbers by decimals	30

*Part 4. Mathematical Induction, Summation Notation,
and Related Topics*

14.1	An example of a proof by mathematical induction	32
1 4.2	The principle of mathematical induction	34
*1 4.3	The well-ordering principle	34
1 4.4	Exercises	35
*I 4.5	Proof of the well-ordering principle	37
1 4.6	The summation notation	37
1 4.7	Exercises	39
1 4.8	Absolute values and the triangle inequality	41
1 4.9	Exercises	43
*I 4.10	Miscellaneous exercises involving induction	44

1. THE CONCEPTS OF INTEGRAL CALCULUS

1.1	The basic ideas of Cartesian geometry	48
1.2	Functions. Informal description and examples	50
*1.3	Functions. Formal definition as a set of ordered pairs	53
1.4	More examples of real functions	54
1.5	Exercises	56
1.6	The concept of area as a set function	57
1.7	Exercises	60
1.8	Intervals and ordinate sets	60
1.9	Partitions and step functions	61
1.10	Sum and product of step functions	63
1.11	Exercises	63
1.12	The definition of the integral for step functions	64
1.13	Properties of the integral of a step function	66
1.14	Other notations for integrals	69

1.15 Exercises	70
1.16 The integral of more general functions	72
1.17 Upper and lower integrals	74
1.18 The area of an ordinate set expressed as an integral	75
1.19 Informal remarks on the theory and technique of integration	75
1.20 Monotonic and piecewise monotonic functions. Definitions and examples	76
1.21 Integrability of bounded monotonic functions	77
1.22 Calculation of the integral of a bounded monotonic function	79
1.23 Calculation of the integral $\int_0^b x^p dx$ when p is a positive integer	79
1.24 The basic properties of the integral	80
1.25 Integration of polynomials	81
1.26 Exercises	83
1.27 Proofs of the basic properties of the integral	84

2. SOME APPLICATIONS OF INTEGRATION

2.1 Introduction	88
2.2 The area of a region between two graphs expressed as an integral	88
2.3 Worked examples	89
2.4 Exercises	94
2.5 The trigonometric functions	94
2.6 Integration formulas for the sine and cosine	97
2.7 A geometric description of the sine and cosine functions	102
2.8 Exercises	104
2.9 Polar coordinates	108
2.10 The integral for area in polar coordinates	109
2.11 Exercises	110
2.12 Application of integration to the calculation of volume	111
2.13 Exercises	114
2.14 Application of integration to the concept of work	115
2.15 Exercises	116
2.16 Average value of a function	117
2.17 Exercises	119
2.18 The integral as a function of the upper limit. Indefinite integrals	120
2.19 Exercises	124

3. CONTINUOUS FUNCTIONS

3.1 Informal description of continuity	126
3.2 The definition of the limit of a function	127

3.3	The definition of continuity of a function	130
3.4	The basic limit theorems. More examples of continuous functions	131
3.5	Proofs of the basic limit theorems	135
3.6	Exercises	138
3.7	Composite functions and continuity	140
3.8	Exercises	142
3.9	Bolzano's theorem for continuous functions	142
3.10	The intermediate-value theorem for continuous functions	144
3.11	Exercises	145
3.12	The process of inversion	146
3.13	Properties of functions preserved by inversion	147
3.14	Inverses of piecewise monotonic functions	148
3.15	Exercises	149
3.16	The extreme-value theorem for continuous functions	150
3.17	The small-span theorem for continuous functions (uniform continuity)	152
3.18	The integrability theorem for continuous functions	152
3.19	Mean-value theorems for integrals of continuous functions	154
3.20	Exercises	155

4. DIFFERENTIAL CALCULUS

4.1	Historical introduction	156
4.2	A problem involving velocity	157
4.3	The derivative of a function	159
4.4	Examples of derivatives	161
4.5	The algebra of derivatives	164
4.6	Exercises	167
4.7	Geometric interpretation of the derivative as a slope	169
4.8	Other notations for derivatives	171
4.9	Exercises	173
4.10	The chain rule for differentiating composite functions	174
4.11	Applications of the chain rule . Related rates and implicit differentiation	176
4.12	Exercises	179
4.13	Applications of differentiation to extreme values of functions	181
4.14	The mean-value theorem for derivatives	183
4.15	Exercises	186
4.16	Applications of the mean-value theorem to geometric properties of functions	187
4.17	Second-derivative test for extrema	188
4.18	Curve sketching	189
4.19	Exercises	191

4.20	Worked examples of extremum problems	191
4.21	Exercises	194
4.22	Partial derivatives	196
4.23	Exercises	201

5. THE RELATION BETWEEN INTEGRATION AND DIFFERENTIATION

5.1	The derivative of an indefinite integral. The first fundamental theorem of calculus	202
5.2	The zero-derivative theorem	204
5.3	Primitive functions and the second fundamental theorem of calculus	205
5.4	Properties of a function deduced from properties of its derivative	207
5.5	Exercises	208
5.6	The Leibniz notation for primitives	210
5.7	Integration by substitution	212
5.8	Exercises	216
5.9	Integration by parts	217
5.10	Exercises	220
*5.11	Miscellaneous review exercises	222

6. THE LOGARITHM, THE EXPONENTIAL, AND THE INVERSE TRIGONOMETRIC FUNCTIONS

6.1	Introduction	226
6.2	Motivation for the definition of the natural logarithm as an integral	227
6.3	The definition of the logarithm. Basic properties	229
6.4	The graph of the natural logarithm	230
6.5	Consequences of the functional equation $L(ab) = L(a) + L(b)$	230
6.6	Logarithms referred to any positive base $b \neq 1$	232
6.7	Differentiation and integration formulas involving logarithms	233
6.8	Logarithmic differentiation	235
6.9	Exercises	236
6.10	Polynomial approximations to the logarithm	238
6.11	Exercises	242
6.12	The exponential function	242
6.13	Exponentials expressed as powers of e	244
6.14	The definition of e^x for arbitrary real x	244
6.15	The definition of a^x for $a > 0$ and x real	245

6.16	Differentiation and integration formulas involving exponentials	245
6.17	Exercises	248
6.18	The hyperbolic functions	251
6.19	Exercises	251
6.20	Derivatives of inverse functions	252
6.21	Inverses of the trigonometric functions	253
6.22	Exercises	256
6.23	Integration by partial fractions	258
6.24	Integrals which can be transformed into integrals of rational functions	264
6.25	Exercises	267
6.26	Miscellaneous review exercises	268

7. POLYNOMIAL APPROXIMATIONS TO FUNCTIONS

7.1	Introduction	272
7.2	The Taylor polynomials generated by a function	273
7.3	Calculus of Taylor polynomials	275
7.4	Exercises	278
7.5	Taylor's formula with remainder	278
7.6	Estimates for the error in Taylor's formula	280
*7.7	Other forms of the remainder in Taylor's formula	283
7.8	Exercises	284
7.9	Further remarks on the error in Taylor's formula. The o-notation	286
7.10	Applications to indeterminate forms	289
7.11	Exercises	290
7.12	L'Hôpital's rule for the indeterminate form $0/0$	292
7.13	Exercises	295
7.14	The symbols $+\infty$ and $-\infty$. Extension of L'Hôpital's rule	296
7.15	Infinite limits	298
7.16	The behavior of $\log x$ and e^x for large x	300
7.17	Exercises	303

8. INTRODUCTION TO DIFFERENTIAL EQUATIONS

8.1	Introduction	305
8.2	Terminology and notation	306
8.3	A first-order differential equation for the exponential function	307
8.4	First-order linear differential equations	308

8.5 Exercises	311
8.6 Some physical problems leading to first-order linear differential equations	313
8.7 Exercises	319
8.8 Linear equations of second order with constant coefficients	322
8.9 Existence of solutions of the equation $y'' + by = 0$	323
8.10 Reduction of the general equation to the special case $y'' + by = 0$	324
8.11 Uniqueness theorem for the equation $y'' + by = 0$	324
8.12 Complete solution of the equation $y'' + by = 0$	326
8.13 Complete solution of the equation $y'' + ay' + by = 0$	326
8.14 Exercises	328
8.15 Nonhomogeneous linear equations of second order with constant coefficients	329
8.16 Special methods for determining a particular solution of the nonhomogeneous equation $y'' + ay' + by = R$	332
8.17 Exercises	333
8.18 Examples of physical problems leading to linear second-order equations with constant coefficients	334
8.19 Exercises	339
8.20 Remarks concerning nonlinear differential equations	339
8.21 Integral curves and direction fields	341
8.22 Exercises	344
8.23 First-order separable equations	345
8.24 Exercises	347
8.25 Homogeneous first-order equations	347
8.26 Exercises	350
8.27 Some geometrical and physical problems leading to first-order equations	351
8.28 Miscellaneous review exercises	355

9. COMPLEX NUMBERS

9.1 Historical introduction	358
9.2 Definitions and field properties	358
9.3 The complex numbers as an extension of the real numbers	360
9.4 The imaginary unit i	361
9.5 Geometric interpretation. Modulus and argument	362
9.6 Exercises	365
9.7 Complex exponentials	366
9.8 Complex-valued functions	368
9.9 Examples of differentiation and integration formulas	369
9.10 Exercises	371

10. SEQUENCES, INFINITE SERIES, IMPROPER INTEGRALS

10.1 Zeno's paradox	374
10.2 Sequences	378
10.3 Monotonic sequences of real numbers	381
10.4 Exercises	382
10.5 Infinite series	383
10.6 The linearity property of convergent series	385
10.7 Telescoping series	386
10.8 The geometric series	388
10.9 Exercises	391
*10.10 Exercises on decimal expansions	393
10.11 Tests for convergence	394
10.12 Comparison tests for series of nonnegative terms	394
10.13 The integral test	397
10.14 Exercises	398
10.15 The root test and the ratio test for series of nonnegative terms	399
10.16 Exercises	402
10.17 Alternating series	403
10.18 Conditional and absolute convergence	406
10.19 The convergence tests of Dirichlet and Abel	407
10.20 Exercises	409
*10.21 Rearrangements of series	411
10.22 Miscellaneous review exercises	414
10.23 Improper integrals	416
10.24 Exercises	420

11. SEQUENCES AND SERIES OF FUNCTIONS

11.1 Pointwise convergence of sequences of functions	422
11.2 Uniform convergence of sequences of functions	423
11.3 Uniform convergence and continuity	424
11.4 Uniform convergence and integration	425
11.5 A sufficient condition for uniform convergence	427
11.6 Power series. Circle of convergence	428
11.7 Exercises	430
11.8 Properties of functions represented by real power series	431
11.9 The Taylor's series generated by a function	434
11.10 A sufficient condition for convergence of a Taylor's series	435

11.11	Power-series expansions for the exponential and trigonometric functions	435
*11.12	Bernstein's theorem	437
11.13	Exercises	438
11.14	Power series and differential equations	439
11.15	The binomial series	441
11.16	Exercises	443

12. VECTOR ALGEBRA

12.1	Historical introduction	445
12.2	The vector space of n-tuples of real numbers.	446
12.3	Geometric interpretation for $n \leq 3$	448
12.4	Exercises	450
12.5	The dot product	451
12.6	Length or norm of a vector	453
12.7	Orthogonality of vectors	455
12.8	Exercises	456
12.9	Projections. Angle between vectors in n-space	457
12.10	The unit coordinate vectors	458
12.11	Exercises	460
12.12	The linear span of a finite set of vectors	462
12.13	Linear independence	463
12.14	Bases	466
12.15	Exercises	467
12.16	The vector space $V_n(\mathbf{C})$ of n-tuples of complex numbers	468
12.17	Exercises	470

13. APPLICATIONS OF VECTOR ALGEBRA TO ANALYTIC GEOMETRY

13.1	Introduction	471
13.2	Lines in n-space	472
13.3	Some simple properties of straight lines	473
13.4	Lines and vector-valued functions	474
13.5	Exercises	477
13.6	Planes in Euclidean n-space	478
13.7	Planes and vector-valued functions	481
13.8	Exercises	482
13.9	The cross product	483

13.10	The cross product expressed as a determinant	486
13.11	Exercises	487
13.12	The scalar triple product	488
13.13	Cramer's rule for solving a system of three linear equations	490
13.14	Exercises	491
13.15	Normal vectors to planes	493
13.16	Linear Cartesian equations for planes	494
13.17	Exercises	496
13.18	The conic sections	497
13.19	Eccentricity of conic sections	500
13.20	Polar equations for conic sections	501
13.21	Exercises	503
13.22	Conic sections symmetric about the origin	504
13.23	Cartesian equations for the conic sections	505
13.24	Exercises	508
13.25	Miscellaneous exercises on conic sections	509

14. CALCULUS OF VECTOR-VALUED FUNCTIONS

14.1	Vector-valued functions of a real variable	512
14.2	Algebraic operations. Components	512
14.3	Limits, derivatives, and integrals	513
14.4	Exercises	516
14.5	Applications to curves . Tangency	517
14.6	Applications to curvilinear motion. Velocity, speed, and acceleration	520
14.7	Exercises	524
14.8	The unit tangent, the principal normal, and the osculating plane of a curve	525
14.9	Exercises	528
14.10	The definition of arc length	529
14.11	Additivity of arc length	532
14.12	The arc-length function	533
14.13	Exercises	535
14.14	Curvature of a curve	536
14.15	Exercises	538
14.16	Velocity and acceleration in polar coordinates	540
14.17	Plane motion with radial acceleration	542
14.18	Cylindrical coordinates	543
14.19	Exercises	543
14.20	Applications to planetary motion	545
14.2	1 Miscellaneous review exercises	549

15. LINEAR SPACES

15.1	Introduction	551
15.2	The definition of a linear space	551
15.3	Examples of linear spaces	552
15.4	Elementary consequences of the axioms	554
15.5	Exercises	555
15.6	Subspaces of a linear space	556
15.7	Dependent and independent sets in a linear space	557
15.8	Bases and dimension	559
15.9	Exercises	560
15.10	Inner products , Euclidean spaces , norms	561
15.11	Orthogonality in a Euclidean space	564
15.12	Exercises	566
15.13	Construction of orthogonal sets. The Gram-Schmidt process	568
15.14	Orthogonal complements . Projections	572
15.15	Best approximation of elements in a Euclidean space by elements in a finite-dimensional subspace	574
15.16	Exercises	576

16. LINEAR TRANSFORMATIONS AND MATRICES

16.1	Linear transformations	578
16.2	Null space and range	579
16.3	Nullity and rank	581
16.4	Exercises	582
16.5	Algebraic operations on linear transformations	583
16.6	Inverses	585
16.7	One-to-one linear transformations	587
16.8	Exercises	589
16.9	Linear transformations with prescribed values	590
16.10	Matrix representations of linear transformations	591
16.11	Construction of a matrix representation in diagonal form	594
16.12	Exercises	596
16.13	Linear spaces of matrices	597
16.14	Isomorphism between linear transformations and matrices	599
16.15	Multiplication of matrices	600
16.16	Exercises	603
16.17	Systems of linear equations	605

16.18 Computation techniques	607
16.19 Inverses and matrices	611
16.20 Exercises	613
16.21 Miscellaneous exercises on matrices	614
Answers to exercises	617
Index	657

Calculus

