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Classification of Lipschitz Mappings

Łukasz Piasecki

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To My Parents

Contents

Introduction

The Lipschitz condition is of great importance in many branches of mathematics. The standard situation is the following: let (M, ρ) be a metric space; we say that a mapping $T : M \to M$ is lipschitzian if there exists a constant k such that, for all $x, y \in M$, we have

$$\rho(Tx, Ty) \leq k\rho(x, y).$$

The class of all mappings satisfying the Lipschitz condition with a constant k is denoted by $L(k)$. The smallest constant k for which the above inequality holds is called the Lipschitz constant for T and is denoted by $k(T)$.

In some approaches, we also investigate a behavior of Lipschitz constants $k(T^n)$ for iterates T^n of T. To describe such behavior, we can find in the literature two special constants:

$$k_\infty(T) = \limsup_{n\to\infty} k(T^n)$$

and

$$k_0(T) = \lim_{n\to\infty} \sqrt[n]{k(T^n)}.$$

The constant $k_\infty(T)$ plays a special role in the fixed point theory for uniformly lipschitzian mappings; we say that a mapping $T : M \to M$ is uniformly lipschitzian if there exists a constant $k \geq 0$ such that, for every $n = 1, 2, \dots$ and all $x, y \in M$, $\rho(T^n x, T^n y) \leq k\rho(x, y)$.

The constant $k_0(T)$, which in case of linear mapping T defined on a Banach space is just the spectral radius of T, has the following interpretation in a general case of lipschitzian mappings:

$$k_0(T) = \inf k_r(T),$$

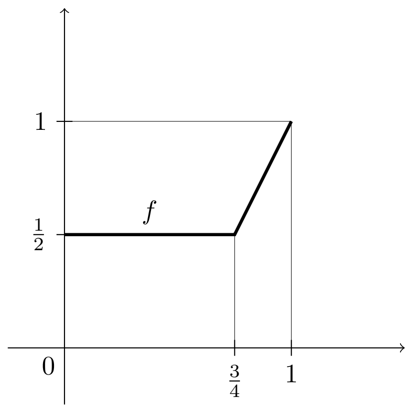
where $k_r(T)$ denotes the Lipschitz constant for T with respect to the metric r, and the infimum is taken over all metrics r that are equivalent to ρ; we say that a metric r is equivalent to ρ if there exist constants $a, b > 0$ such that, for all $x, y \in M$, $a\rho(x, y) \leq r(x, y) \leq b\rho(x, y)$.

In general, it is hard to determine or give nice evaluations for the constants $k_0(T)$ and $k_\infty(T)$ because it demands nice estimates of $k(T^n)$. The possible growth of the sequence $k(T^n)$ can be regulated by the following standard inequality:

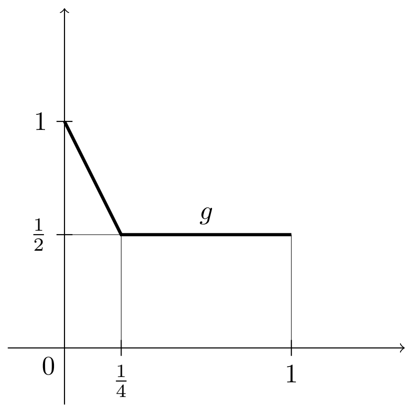
$$k(T^n) \leq k(T)^n \text{ for } n = 1, 2, \dots,$$

which, in case of any class $L(k)$, is sharp. However, in many cases, this estimate doesn't give good approximation of $k(T^n)$. The sequence $\{k(T^n)\}$ may behave very unpredictably even in simple situations of functions defined on the interval $[0,1]$, as seen below.

Let us consider two "similar" graphs. The first graph describes the function f:



It is easy to check that $k(f) = 2$ and $k(f^n) = 2^n$ for $n = 1, 2, \dots$. However, if we slightly change the graph of function f, then the situation will totally change:



In this case, we can divide the interval $[0,1]$ into two regions: $[0,1/4]$ and $(1/4,1]$. It is easy to verify that the function g extends a distance between any two points in $[0,1/4]$, $|g(s) - g(t)| = 2\,|s - t|$, sending them to the interval

$(1/4, 1]$ on which the second iterate contracts it back, sending them to $\left\{\frac{1}{2}\right\}$. Consequently, $k(g) = 2$, but $k(g^n) = 0$ for $n \geq 2$.

In the above samples, we have $k(f) = k(g) = 2$. Hence, $f, g \in L(2)$. Nevertheless, remaining iterates behave in various ways: $k_\infty(f) = \infty$, $k_0(f) = 2$, and $k_\infty(g) = 0$, $k_0(g) = 0$.

In this book we deal with a problem of more precise classification of lipschitzian mappings. It seems natural that a condition that describes such classes should satisfy several principles. It should regulate the possible growth of the sequence of Lipschitz constants $k(T^n)$, as well as ensure nice estimates for $k_0(T)$ and $k_\infty(T)$. We also expect that such condition will provide some new results in the metric fixed point theory. The last and the most important request is that it has to be relatively easy to check.

In this monograph, we widely study the *mean Lipschitz condition*, which was introduced by Goebel, Japón Pineda, and Sims: suppose that $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $n \geq 1$, $\alpha_i \geq 0$, $\alpha_1 > 0$, $\alpha_n > 0$, and $\sum_{i=1}^n \alpha_i = 1$; we say that $T : M \to M$ is α-lipschitzian for the constant $k \geq 0$ if, for all $x, y \in M$,

$$\sum_{i=1}^{n} \alpha_i \rho(T^i x, T^i y) \leq k\rho(x, y).$$

The class of all mappings that satisfies the above inequality with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and a constant $k \geq 0$ is denoted by $L(\alpha, k)$.

We also focus on mean Lipschitz condition described by averages of order $p \geq 1$: let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be as above and $p \geq 1$; we say that a mapping $T : M \to M$ is (α, p)-lipschitzian for the constant $k \geq 0$ if, for all $x, y \in M$,

$$\left(\sum_{i=1}^{n} \alpha_i \rho(T^i x, T^i y)^p \right)^{1/p} \leq k\rho(x, y).$$

The class of all mappings that satisfies the above condition with $\alpha = (\alpha_1, \ldots, \alpha_n)$, $p \geq 1$ and $k \geq 0$ is denoted by $L(\alpha, p, k)$. Let us note that, for $p = 1$, we get a definition of α-lipschitzian mapping for the constant k. We usually write $L(\alpha, k)$ instead of $L(\alpha, 1, k)$. It is clear that, for $n = 1$, we get nothing more than the classical definition of k-lipschitzian mapping. Hence, in this special case, the mean Lipschitz condition determines the class $L(k)$.

The presented monograph arose as a result of discussions that took place during a seminar under the supervision of Professor Kazimierz Goebel at the University of Maria Curie-Sklodowska in Lublin, Poland. The included results have been presented many times during international conferences as well as at seminars devoted to the metric fixed point theory. Many of them have been already published. A part of the presented material appeared only in a PhD thesis of the present author [70] written under the guidance of Professor Kazimierz Goebel.

The book is addressed to advanced undergraduate and graduate students as well as to professionals looking for new topics in the metric fixed point theory. Nevertheless, we deeply believe that this monograph will be of interest among readers working on other areas (for example, differential equations and dynamical systems). The proposed text is self-contained, and only a basic knowledge of functional analysis and topology is required.

The monograph covers approximately 220 pages of a systematic course, with updated facts concerning discussed topics, with accompanying illustrations, a rich collection of examples, and open problems. The book does not include typical exercises. Instead, we can find many sentences like “It is easy to see...”, “We leave to the reader...”,“Observe that..” and so forth.

This book does not aim to compete with any book devoted to the metric fixed point theory. Our goal is to provide a systematic, self-contained course of a new classification of Lipschitz mappings, pointing out its application in many topics of the metric fixed point theory. Moreover, we encourage new adept to acquaint themselves with some existing books, in particular [49], [36] and [31], which, with no doubt, could be useful for further development of this new field.

I am deeply grateful to Professor Kazimierz Goebel for creating the atmosphere conducive to writing this book. His constant interest in progress, support, and discussions while writing were valuable to me and undoubtedly enriched the content of this monograph considerably.

It is my kind duty to acknowledge Professors Tomás Domínguez Benavides, Emanuele Casini, and Stanisław Prus for pointing out errors and omissions as well as for valuable remarks that improved this book.

I would like to express my deep sense of gratitude to my colleague Dr. Piotr Oleszczuk for his kind support in proofreading and verifying the substantial content of subsequent parts of the text.

I would like to express special thanks to my friend and scientific partner, Dr. Víctor Pérez García. During his postdoctoral position at Maria Curie-Sklodowska University, we had a number of opportunities to hold discussions as well as intensive and fruitful cooperation. Many results obtained then were included in this monograph.

I am thankful to Sunil Nair for excellent cooperation on publishing this book.

Chapter 1

The Lipschitz condition

Let (M,ρ) be a metric space. By $B_M(x,r)$, we denote *the closed ball* centered at $x \in M$ with radius $r > 0$:

$$B_M(x,r) = \{y \in M : \rho(x,y) \leq r\}.$$

By $S_M(x,r)$, we denote *the sphere* centered at $x \in M$ with radius $r > 0$:

$$S_M(x,r) = \{y \in M : \rho(x,y) = r\}.$$

As usual, we shall drop the subscript when M is clear from the context. For a subset A of (M,ρ), $\operatorname{diam}(A)$ denotes *the diameter* of A; that is,

$$\operatorname{diam}(A) = \sup\{\rho(x,y) : x, y \in A\}.$$

A subset $A \subset M$ is called *bounded* if $\operatorname{diam}(A) < \infty$.
We say that $z \in M$ is a *fixed point* of mapping $T : M \to M$ if $z = Tz$. The set of all fixed points of T is denoted by $\operatorname{Fix}(T)$.

A mapping $T : M \to M$ is said to be *lipschitzian* if there exists $k \geq 0$ such that, for all $x, y \in M$

$$\rho(Tx, Ty) \leq k\rho(x,y). \tag{1.1}$$

If we want to indicate $k \geq 0$, then we refer to such a mapping as *k-lipschitzian.* By $L(k)$, we denote the class of all lipschitzian mappings that satisfies (1.1) with a constant k. It is clear that lipschitzian mappings are uniformly continuous. For a given mapping T, the smallest k for which (1.1) holds (such k always exists) is called the *Lipschitz constant* for T and will be denoted by $k_\rho(T)$ or simply $k(T)$ when the underlying metric is clear from the context. A mapping $T : M \to M$ is said to be a *contraction* if $k(T) < 1$. If $k(T) \leq 1$, then T is called *nonexpansive*. Thus, the class of nonexpansive mappings includes all contractions and isometries, in particular, the identity.

1.1 Nonlinear spectral radius

For two lipschitzian mappings $T, S : M \to M$ and all $x, y \in M$, we have

$$\rho(TSx, TSy) \leq k(T)\rho(Sx, Sy) \leq k(T)k(S)\rho(x,y)$$

and so

$$k(T \circ S) \leq k(T)k(S). \tag{1.2}$$

In particular, (1.2) regulates the possible growth of the sequence of Lipschitz constants for iterates T^n of T,

$$k(T^{n+m}) = k(T^n T^m) \leq k(T^n) \cdot k(T^m) \text{ for } m, n = 1, 2, \ldots$$

and consequently

$$k(T^n) \leq k(T)^n \text{ for } n = 1, 2, \ldots. \tag{1.3}$$

Thus, if T is nonexpansive, then all its powers T^n are also nonexpansive. If T is a contraction, then $\lim_{n\to\infty} k(T^n) = 0$. To recall another basic observation about the sequence of Lipschitz constants $\{k(T^n)\}$ of a given mapping T, we shall need the following well-known fact:

Lemma 1.1 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers satisfying, for all $m, n = 1, 2, \ldots,$ the following condition:*

$$a_{m+n} \leq a_m \cdot a_n. \tag{1.4}$$

Then, $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \inf \left\{ \sqrt[n]{a_n} : n = 1, 2, \ldots \right\}$.

Proof. Put

$$a = \inf \left\{ \sqrt[n]{a_n} : n = 1, 2, \ldots \right\}.$$

Fix $\epsilon > 0$. There is $k \in \mathbb{N}$ such that $a \leq \sqrt[k]{a_k} \leq a + \epsilon$. For any positive integer n, we can write

$$n = qk + r,$$

where q and r are nonnegative integers with $0 \leq r \leq k - 1$. Then, using (1.4), we get

$$a_n \leq a_{qk} \cdot a_r \leq a_k^q \cdot a_r.$$

Thus,

$$\sqrt[n]{a_n} \leq a_k^{q/n} \cdot a_r^{1/n} \leq (a + \epsilon)^{kq/n} \cdot a_r^{1/n}.$$

Letting n tend to infinity and for fixed k we obtain

$$\lim_{n\to\infty} \frac{kq}{n} = \lim_{n\to\infty} \left(1 - \frac{r}{n}\right) = 1,$$

and consequently,

$$a \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq a + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this implies $\limsup\limits_{n\to\infty} \sqrt[n]{a_n} = a$. On the other hand, $\sqrt[n]{a_n} \geq a$ for each n and so $\liminf\limits_{n\to\infty} \sqrt[n]{a_n} \geq a$. Thus, $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = a$.

$\square$

We have already seen that the sequence of Lipschitz constants $\{k(T^n)\}$ satisfies (1.4). Thus, the constant

$$k_0(T) = \lim_{n\to\infty} \sqrt[n]{k(T^n)}$$

is well defined and

$$k_0(T) = \lim_{n\to\infty} \sqrt[n]{k(T^n)} = \inf\left\{ \sqrt[n]{k(T^n)} : n = 1, 2, \dots \right\}. \tag{1.5}$$

Using (1.3), we obtain the following evaluation for $k_0(T)$:

$$k_0(T) = \lim_{n\to\infty} \sqrt[n]{k(T^n)} \leq k(T).$$

We say that two metrics d and ρ are *equivalent* if there exist two constants $a > 0$ and $b > 0$ such that, for all $x, y \in M$

$$ad(x,y) \leq \rho(x,y) \leq bd(x,y).$$

It is easy to check that any mapping T, which is lipschitzian with respect to a given metric ρ is also lipschitzian with respect to any equivalent metric d. The respective Lipschitz constants $k_\rho(T)$ and $k_d(T)$ may differ and the difference is regulated by the relation

$$d(Tx,Ty) \leq \frac{1}{a}\rho(Tx,Ty) \leq \frac{1}{a}k_\rho(T)\rho(x,y) \leq \frac{b}{a}k_\rho(T)d(x,y).$$

This implies that

$$k_d(T) \leq \frac{b}{a}k_\rho(T).$$

Changing roles of metrics d and ρ, we obtain

$$k_\rho(T) \leq \frac{b}{a}k_d(T).$$

Obviously, the same holds for all iterates T^n of mapping T; hence,

$$\frac{a}{b}k_\rho(T^n) \leq k_d(T^n) \leq \frac{b}{a}k_\rho(T^n). \tag{1.6}$$

In particular, (1.6) implies that the constant $k_0(T)$ is independent of the selection of a metric d, which is equivalent to ρ. Thus, using (1.5), for each equivalent metric d, we have $k_0(T) \leq k_d(T)$. As a consequence, $k_0(T) \leq \inf k_d(T)$, where infimum is taken over all metrics d, which are equivalent to ρ. Actually, we have $k_0(T) = \inf k_d(T)$. Indeed, for arbitrary $\epsilon > 0$ set $\lambda = 1/(k_0(T) + \epsilon)$. Let us define a metric d_λ by putting for each $x, y \in M$

$$d_\lambda(x,y) = \sum_{i=0}^{\infty} \rho(T^i x, T^i y)\lambda^i = \rho(x,y) + \sum_{i=1}^{\infty} \rho(T^i x, T^i y)\lambda^i. \tag{1.7}$$

It is easy to observe that the metric d_λ is equivalent to ρ and

$$\rho(x,y) \leq d_\lambda(x,y) \leq \left(\sum_{i=0}^{\infty} k_\rho(T^i)\lambda^i\right)\rho(x,y). \tag{1.8}$$

Now, for all $x, y \in M$, we have

$$\begin{aligned} d_\lambda(Tx,Ty) &= \sum_{i=0}^{\infty}\rho(T^{i+1}x, T^{i+1}y)\lambda^i \\ &= \frac{1}{\lambda}\sum_{i=0}^{\infty}\rho(T^{i+1}x, T^{i+1}y)\lambda^{i+1} \\ &\leq \frac{1}{\lambda}\sum_{i=0}^{\infty}\rho(T^{i}x, T^{i}y)\lambda^{i} \\ &= \frac{1}{\lambda}d_\lambda(x,y). \end{aligned}$$

Thus,

$$k_{d_\lambda}(T) \leq 1/\lambda = k_0(T) + \epsilon. \tag{1.9}$$

Since $\epsilon > 0$ is arbitrarily chosen, we finally obtain the following characterization of $k_0(T)$:

$$k_0(T) = \inf\left\{k_d(T) : \ d \text{ is equivalent to } \rho\right\}. \tag{1.10}$$

1.2 Uniformly lipschitzian mappings

Of particular interest in the metric fixed point theory is a class of *uniformly lipschitzian mappings*. We say that a mapping $T : M \to M$ is *uniformly lipschitzian*, if there exists a constant $k \geq 0$ such that for all $x, y \in M$ and $n = 1, 2, \ldots$, we have

$$\rho(T^n x, T^n y) \leq k\rho(x,y).$$

If we want to indicate $k \geq 0$, then we refer to such mapping as a *uniformly k-lipschitzian*. Equivalently, a lipschitzian mapping T is uniformly lipschitzian, if and only if

$$\sup\left\{k(T^n) : n = 1, 2, \ldots\right\} < \infty. \tag{1.11}$$

We shall denote the class of all uniformly k-lipschitzian mappings by $L_u(k)$.

It is well known that there exists another characterization of such mappings. It is clear that, if for some equivalent metric d we have $k_d(T) \leq 1$, then $k_\rho(T^n) \leq b/a$ for $n = 1, 2, \ldots$ (a and b are taken from the definition of equivalent metrics). Hence, condition (1.11) is satisfied. On the other hand, if T is uniformly lipschitzian, then it is nonexpansive with respect to the equivalent metric defined by

$$d(x,y) = \sup \{\rho(T^n x, T^n y) : n = 0,1,2,\ldots\}. \tag{1.12}$$

Indeed, for all $x, y \in M$, we have

$$\rho(x,y) \le d(x,y) \le \sup\{k(T^n) : n = 0,1,2,...\}\, \rho(x,y)$$

and

$$\begin{aligned} d(Tx,Ty) &= \sup\left\{\rho\left(T^{n+1}x, T^{n+1}y\right) : n = 0,1,2,\ldots\right\} \\ &\le \sup\{\rho(T^n x, T^n y) : n = 0,1,2,\ldots\} \\ &= d(x,y). \end{aligned}$$

Finally, we conclude that a mapping T is uniformly lipschitzian if and only if there exists an equivalent metric with respect to which T is nonexpansive. Consequently, for any uniformly lipschitzian mapping T, we have $k_0(T) \le 1$.

Also, there is another useful constant that appears during studies devoted to uniformly lipschitzian mappings. For any lipschitzian mapping $T : M \to M$, we define the constant $k_\infty(T)$ as

$$k_\infty(T) = \limsup_{n\to\infty} k(T^n).$$

Obviously, a mapping T is uniformly lipschitzian if and only if $k_\infty(T) < \infty$. Usefulness of the constant $k_\infty(T)$ follows from the fact that usually

$$k_\infty(T) < \sup\{k(T^n) : n = 1,2,\ldots\}.$$

We only mention that all the above definitions carry over to a Banach space setting $(X, \|\cdot\|)$ by taking $\rho(x,y) = \|x-y\|$. It is easy to verify that, for any nonempty subset C of a Banach space X and two lipschitzian mappings $T, S : C \to X$, we additionally have the following elementary inequalities: $k(T+S) \le k(T) + k(S)$ and $k(\lambda T) = \lambda k(T)$ for $\lambda \ge 0$.

In this text, unless otherwise stated, we always assume that all considered sets are nonempty.